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STUDIES IN CIRCLE GEOMETRY*

BY

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1. The Thesis proper consists of the following papers:—

(i) On Quadratic Equations: *The Mathematics Student*, Vol. IV (1936).

(ii) On the Group of the Miquel-Clifford Configuration: *Annamalai Univ. Jour.* Vol. I (1932).

(iii) On certain Cremona transformations in circle-space connected with the Miquel-Clifford Configuration: *Proc. of the Camb. Phil. Soc.* XXXIII (1937).

(iv) On the Miquel-Clifford Configuration: *Proc. Ind. Acad. Sc.* VII (1938).

(v) The Miquel-Clifford Configuration in the Geometries of Mobius and Laguerre: *Annamalai University Journal*, VII (1938).

(vi) On a principle of duality in circle, sphere and line-Geometries: *Mathematische Zeitschrift* Bd. 44 (1938).

(vii) Studies in Turbine Geometry—I: The concepts of Turbine Geometry *Jour. of the Indian Math. Soc.* III (1938).

(viii) Studies in Turbine Geometry—II: On the sub-geometries of Lie which belong to the Mobius Laguerre pencil: *Proc. Ind. Acad. Sc.* VIII (1938).

(ix) Studies in Turbine Geometry—III: The non oriented line element in two dimensional Mobius Geometry: *Proc. Ind. Acad. Sc.* IX (1939).

(x) Studies in Turbine Geometry—IV: The Topology of oriented and non oriented line elements in the inversive plane: [To appear shortly in the *Proc. Ind. Acad. Sc.*]

* [This paper formed the "Introduction" to a thesis with the above title accepted by the Madras University for its Doctor of Science degree. As doctorate theses are not readily available for consultation in India the Editor has reluctantly allowed its publication though he feels that the standard is somewhat higher than that which *The Mathematics Student* has in view.—ED.]

A selection of 13 other published papers representative of other fields in which work was done at different times have been submitted in support of the thesis.

2. One-dimensional Circle-Geometry

The circle in a Euclidean space of one dimension is a pair of points, the centre being their middle point, and the radius half the distance between them. Hence the Mobius Group of circle transformations carrying circles into circles and circles of zero radius into circles of zero radius becomes identical, in this case, with the projective group of the ambient space.

We represent the point pair t_1 t_2 which are roots of $xt^2 - 2yt + z = 0$ by the point $x : y : z$ of a projective plane, the coincident point-pairs (circles of zero radius) corresponding to points on the conic $\Omega : xz - y^2 = 0$. The homographic group of the line now induces on the projective plane collineations which leave Ω invariant. Thus the geometry of point pairs on a line is found to be identical in structure with the geometry of a hyperbolic plane with Ω for its Cayley Absolute, both real and ideal points being taken into account. If the line be treated as Euclidean *i.e.* endowed with a parabolic metric defined by a point at infinity (P_∞), we have not only the notion of centre and radius but may also define the analogue of the straight line as a point pair of which one member is P_∞. The contact of two point pairs would imply the coincidence of one of their points which leads to the somewhat startling result that every circle in 1-dimensional space has only two tangents!

If we differentiate between the orders t_1 t_2 and t_2 t_1 of the point pair, we are dealing with oriented circles in 1-space. The roots being individually known may be subjected to independent linear transformations and this constitute a double-binary domain topologically equivalent to the surface of a ruled quadric, the analytical mechanism for effecting this correspondence being provided by a redundant co-ordinate. These topics and the adjunction of ideal elements to secure closure are discussed in the first part of paper (i).

The geometry of point pairs is of interest because it stands at the junction of two important mathematical disciplines when we pass on to spaces of higher dimensions according as we emphasise the binary or the quadratic character of the binary quadratic. One road —the one we are not pursuing—leads to

3. The Geometry of the Binary n-ic

under the group of homographic transformations. The algebraic part of this theory dates from the time of Cayley and Sylvester and was the subject of considerable activity in the hands of Clebsch, Gordon, Aronhold and others. On the geometric side the invariant theory of the binary n-ic is found to be identical—in the sense of Klein—with the geometry of a projective n-space under the group of collineations which carry a particular rational norm curve into itself. Not only has the geometrical picture been used (by Meyer, Stephanos and White to mention a few names) for obtaining a representation with respect to the norm curve of invariants, covariants and of linear systems of binary forms, but it has also been used the other way by Comessatti to work out a geometrical theory of binary forms. A series of striking results by Vaidyanathaswamy and Ramamurti on the study of correlations in relation to the norm curve and its connection with the theory of double-binary equiforms and of quadrics and simplexes related to the norm curve bear witness to the richness of a field which is yet to receive adequate attention.

The study of oriented n-ics appears to have received no attention at all. Orientation implies that we distinguish between certain permutations of the n roots $t_1\ t_2 \ldots\ t_n$ but regard others as equivalent: hence the number of possible orientations is equal to the number of sub-groups of the permutation group on n symbols. In the later part of paper (i) I have tried to indicate roughly some features of geometry of the binary cubic both oriented and non-oriented.

The other road—the one we are to pursue—leads from the Euclidean 2-point to the geometry of circles and spheres in n-dimensional Euclidean Space, and we begin with

4. Circle Geometry in 2 dimensions.

Because of the ease with which it can be drawn, its perfect symmetry and its many applications even in a primitive society, the circle must have been a familiar geometrical object from the very infancy of the human race. The Greeks who were lovers of the beautiful and of the abstract and to whom the world owes the logical development of geometry from simple premises, discovered the well-known properties of the circle which have been codified in Euclid's *Elements*. So completely had the circle captured the hearts of Aristotle, Hipparchus and other astronomers right up to the time

of Kepler that they could not conceive of the Creator being so unaesthetic as to use figures other than the circle in his plan of the Universe! The study of sections of a cone, again a product of the genius of the Greeks, assigned the circle to the family of conic sections—a position which gained greater significance with the advent of Cartesian Geometry in the 17th century. The rapid development of projective geometry in the 19th century after a long interval of neglect and the introduction of the circular points at infinity and of its 3-dimensional analogue the imaginary circle at infinity, by Poncelet, made clear the nature of the specialisation of the conic and of the quadric which results in the circle and the sphere. In 1853 young Laguerre a pupil of Chasles, then 19 years old and a student at the Ecole Polytechnique, gave a projective definition of the Euclidean angle to be followed six years later by Cayley's brilliant generalisation expressing all metrical magnitudes as projective relations of the figure to an "absolute" conic or quadric in such a way that when the latter degenerated into Poncelet's point-pair or imaginary circle at infinity the magnitudes were identical with those of Euclidean geometry. It thus became possible to treat circles in a plane by purely projective methods as conics through two fixed points or as stereographic projections of sections of a quadric from a fixed point thereon—the view adopted in Baker's monumental "*Principles of Geometry*".

5. The ideas of Plücker and Klein

Of the rich harvest of ideas which are the legacy of the 19th century the two which concern us most are firstly, the development of the Group concept and its use as a basis for the classification of geometries by Felix Klein, and secondly Plücker's idea that space may be considered not only as an aggregate of points but also of geometrical objects of any kind, the dimension of the space being dependent on the choice of the basic element.

Klein's definition of a geometry as the study of those properties of geometrical objects which are unaltered by the transformations of a particular group given in his Erlanger Program of 1872 not only enabled Euclidean and non-euclidean geometries to be imbedded into projective geometry as sub-geometries thereof but threw fresh light on the significance of Cayley's Absolute in classical non-euclidean geometry. The ideas of the Erlanger Program have been the inspiration behind a large amount of geometrical researches for nearly half a century particularly by way of the creation of new

geometries and of the interpretation of Elementar-mathematik vom hoheren Standpunkt aus. The interpretation of the configurations of one geometry from the standpoint of another with a different transformation group is one of the most fascinating fields opened up by Klein, a region in which little work has been done yet.* It must, however, be stated that there are important geometrical schemes which cannot be squeezed into the Erlanger framework and that a more comprehensive basis of classification is being studied under which geometry becomes a study of a "geometrical object" with components. †

Plücker's representation of lines in a 3-space by points on a quadric in 5-dimensions has given a new and modern meaning to the term "circle geometry", which no longer means the investigation of the geometrical properties of one or more circles, but a study of the structural peculiarities of the 3-dimensional manifold of circles in a plane. The structure imposed on the manifold does not arise merely out of the fact that our basic element is the circle but will be determinate only when the transformation group has been specified, that is, only when we have decided what differences we shall ignore and what differences we shall recognise. Our vision now comprehends the totality of all circles in a plane, these being represented by points in a manifold in higher space and we can see in the structure of the manifold the imprint left by our transformation group. Among the transformation groups of circle geometry which have been so far studied are those associated with the names of Möbius, Laguerre and Lie.

6. Möbius Geometry

August Ferdinand Möbius was the most gifted of Gauss' pupils and his name is associated with several concepts which are recognised to be of first rate importance today, for example homogeneous coordinates, the one-sided Möbius band, and Möbius Null-system to name only a few. Klein refers to him as the "eigentlichen Vorlaufer des Erlanger Programms." Möbius deals with non-oriented circles, and the Möbius Group comprises all transformations of circles into circles in which point circles go over into point circles. The

* A very interesting example of such a study that I have come across is that of the isogonal transformation associated with a triangle in a Euclidean plane reinterpreted in terms of the ideas of Möbius-geometry: Vaidyanathaswamy: A memoir on cubic transformations associated with a desmic system, *Jour. Ind. Math. Soc.* Vol. 17 Suppl.

† Schouten and Haantjes: on the Theory of the Geometric object, *Proc. Lond. Math. Soc.* Vol 42.

inspiration for this study must have come from his master's introduction of the complex variable $z=x+iy$ into analysis; for one of the simplest methods of obtaining the group in question is to represent points in the Gauss plane by means of the complex variable and subject them to the homographic transformations $z'=(az+b)/(cz+d)$ or $z'=(a\bar{z}+b)/(c\bar{z}+d)$. The real transformations form a mixed 6 parameter group consisting of two continuous systems of transformations. From our point of view the group may be defined as the result of inversion in one or more circles, one system corresponding to an even number of inversions and the other to an odd number, the former constituting a sub-group of the Möbius group. To secure (1, 1) correspondence it is necessary to adjoin an ideal "point at infinity" to the Euclidean plane and, if imaginary elements are also taken into account, two imaginary ideal lines through it. The inversive plane may thus be obtained from the projective plane by contracting, as it were, the line at infinity into a point and drawing out the circular points into lines through this point. The geometrical mechanism for effecting this change of closure is stereographic projection and accordingly we find that the inversive group may be represented by projective transformations in 3-space which leave a sphere invariant. In this representation each circle in the plane is represented by the plane whose section of the sphere projects into the circle in question, or if we prefer it, by its pole with respect to the sphere. In this imbedding of inversive geometry of the plane into the projective geometry of space, the two systems of transformations correspond to collineations which carry each of the two systems of generators of the sphere into itself or interchange the two systems. It is also clear that the inversive group of the plane may be identified with the transformation group of the projective plane treated as a double binary domain, by means of isotropic coordinates, when the binary variables are treated as digredient.

Möbius geometry in Euclidean spaces of higher dimensions exhibits practically all the features of the 2-dimensional case except that since a quadric in n-dimensions has two systems of generating regions only when n is odd, the distinctions between the two types of transformations will arise only for inversive geometry in an even number of dimensions. In fact we shall meet with examples later which go to show that circle geometry is highly sensitive to the residue modulo 2 of the dimensional number.

7. The Geometries of Laguerre and Lie

To Edmond Laguerre the French mathematician who has been already mentioned in connection with the projective definition of an angle, we owe another fruitful idea—that of the oriented line (semi-droit) and of the oriented circle. In a series of papers under the name "geometrie de direction" beginning from 1879 he developed this idea and discovered an analogue for Möbius inversion in his "transformation par directions reciproques", or Laguerre-inversion as it is called today. He discovered the invariants under Laguerre inversion of the figure formed by 4 oriented lines, just as Mobius had discussed the geometrical significance of the cross ratio of 4 complex numbers in the Gauss plane. In fact Laguerre geometry may be considered as the dual of Möbius geometry with the oriented line as the basic element, the transformations defining the geometry in Klein's sense being those which carry oriented lines into oriented lines and oriented circles as proper envelopes of oriented lines into oriented circles.

The effect of orienting the circles is to cover the 3-dimensional projective space of Möbius twice over, so that the manifold of all oriented circles becomes one whose fundamental group is the identity, that is, a 3-sphere. An actual representation of oriented circles in a plane by points on a quadric in 4-space, and more generally of oriented spheres in a euclidean n-space by points on a quadric in $[n+2]$ is obtained by the introduction of the signed radius as an additional coordinate, opposite signs corresponding to opposite orientations. We have then $n+3$ homogeneous coordinates connected by a quadratic identity and hence a representation on the *Lie Quadric* Ω of a projective $[n+2]$. The Laguerre group now turns out to be the group of projective transformations in $[n+2]$ which carry Ω into itself, and keep a particular point I thereof called the "point sphere at infinity" invariant. (vide paper vii p. 100).

There is a close parallelism between the Möbius and Laguerre sphere geometries in a Euclidean $[n]$ since both contain $(n+1)(n+2)/2$ parameters. In two dimensions we may exhibit this parallelism by the introduction of the "dual numbers" which appear to have been first used by Clifford but have become familiar to us by the work of Scheffers and Study. These are numbers of the form $a+\varepsilon b$ and are added and multiplied in the usual way except that we take $\varepsilon^2=0$. We may introduce as the coordinates of the oriented line $x\cos\alpha + y\sin\alpha = p$, the dual number $\alpha+\varepsilon p$ so that in

Mobius Geometry	**Laguerre Geometry**
the point (x, y) is pictured by the complex number $x + iy = z$	the line (α, p) is pictured by the dual number $\alpha + \varepsilon p = u$

The transformations of the two groups may be represented by

$z' = (az + b)/(cz + d)$ where a, b, c, d are *complex numbers.*	$u' = (au + b)(cu + d)$ where a, b, c, d, are *dual numbers.*

These transformations may be considered as contact transformations which carry

non-oriented line elements through the same point	oriented line elements on the same oriented line

into other line elements of the same kind in such a way that

the angle between two line elements through a common point	the distance between two line† elements on the same oriented line

remains invariant. The general

conformal group	equilong group

of the plane may be represented by the transformation

$z' = f(z)$	$u' = f(u)$

at all places where f is an analytic function.

However, the analogy breaks down at a certain point since, when represented as a sub-group of the projective group in $[n+2]$ with an invariant quadric Ω, the first carries a proper section of Ω into itself while the latter carries a tangential section of Ω into itself. This difference shows itself, for example, in the existence of a system of similarly oriented parallel lines which are all determinate when one of them is given, a situation for which there is no analogue in Möbius Geometry.

With the Norwegian Mathematician Sophus Lie we come nearer to our own times for many of the investigations initiated by him such as his work on continuous groups, on differential equations and on contact transformations have considerable interest at the present day. He was in close contact with two of the greatest geometers of his day—Felix Klein and Gaston Darboux—and to him we owe the study of transformations of oriented circles* or spheres into oriented circles or spheres so as to conserve proper contact. In other words

† This is the "engere" Laguerre Group. The wider group will change all tangential segments in the same ratio.

* The oriented straight line and plane have no special status in Lie's geometry but are merely circles or spheres of infinite radius.

the Lie group of Höhere Kugelgeometrie may be considered as consisting of transformations of oriented surface elements in such a way that spheres are carried over into spheres. Since oriented spheres in $[n]$ correspond to the points on a quadric Ω in $[n+2]$, this group is isomorphic with the automorphic group of collineations of Ω in a projective $[n+2]$. Möbius geometry corresponds to the sub-group which keeps a particular point m and therefore also its polar prime w.r.t. Ω invariant, while Laguerre geometry may also be imbedded in Lie's as the sub-group which keeps another point l invariant, this point lying on the quadric (vide paper (vii) pp. 98-101). I have considered in paper (viii) a pencil of geometries obtained by taking as the invariant prime any member of the pencil formed by the 2 primes which are respectively invariant in the Mobius and Laguerre geometries. The points in $[n+2]$ other than those on Ω have had no geometrical significance attached to them. In my last 4 papers I have interpreted them as "turbines" and tried to study their mutual relations and to use them as tools in our study of Sphere geometry.

One of the most interesting correspondences which Lie worked out is one between oriented spheres in [3] and lines in [3] so that spheres which touch correspond to intersecting lines. The introduction of the turbine concept gives this a richer significance as one in which a linear complex corresponds to a turbine and two complexes in involution to "conjugate turbines". Special linear complexes (straight lines) correspond to special turbines (oriented spheres).

8. The Miquel Clifford Configuration

which is the subject of study in papers (ii), (iii), (iv), (v) belongs to Möbius geometry. Three lines (1), (2), (3) have 3 intersections (12), (23), (31) which lie on the circumcircle (123). With 4 lines we have 4 such circles which Wallace in 1804 and Steiner in 1827 proved to be concurrent. The result may have been known earlier since it follows from the property discovered by Lambert in 1761 that the circumcircle of a triangle formed by 3 tangents to a parabola passes through the focus. In 1836 Miquel proved that the points of concurrence of the 5 quadrilaterals derived from 5 lines are concyclic and finally Clifford in 1870 showed that the alternation of concurrent circles and concyclic points persists indefinitely. Meanwhile Miquel showed that his own theorem concerning 5 lines remains true if the lines are replaced by concurrent circles, a result obviously deducible by inversion; but Miquel was unaware of the

inversive transformation though Plucker had used it in 1831 in *Crelle's Journal* (Bd. 11), and Miquel's proof deals directly with circles.

As far as I am aware it appears to be G. H. Grace and later E. H. Neville that first inverted the Miquel Clifford configuration and showed that the resulting configuration gained in symmetry, for it now consisted of 2^{n-1} circles and 2^{n-1} points such that n circles pass through each point and n points lie on each circle. The configuration is homogeneous in the sense that each of the points of multiple incidence by which the configuration is characterised stands on the same footing and the figure may be generated from the n circles concurrent thereat. These symmetries are lost in Clifford's original configuration of points lines and circles, thus showing how a configuration suffers when transplanted into foreign geometry.

In his paper referred to above † Grace showed that the Miquel Clifford configuration has no analogue in [3] but has one in [4], while there are configurations in [3] which have no analogue in the plane — another example to showing how sensitive circle geometry is to the residue modulo 2 of the dimensional number.

In paper (ii) I have considered the group of permutations of the points and circles in the M. C. C. (Miquel Clifford configuration) which preserves all the incidence relations. There are two such automorphic groups according as we allow points to be replaced by points and circles by circles or place no restriction whatever. Both the groups are shown to be Abelian groups generated by permutable operations each of period 2.

If the number of circles n concurrent at O and generating the M. C. C. is even there is an associated Miquel Point M which is the last to be obtained in the series. An additional circle through O gives rise to an additional circle through M namely the Clifford circle of the $n+1$ circles through O. Paper (ii) studies the transformation in circle space which carries the one into the other. It is shown to be a Cremona Transformation of the De Jonquiere type whose F-elements of unit multiplicity are the given circles and the element of multiplicity $n-2$ the point circle at O. Clifford n-fold parabolas are shown to give an alternative approach leading to the same result.

† Vide a remarkable paper by J. H. Grace on "Circles, Spheres and Linear Complexes", *Proc. Camb. Phil. Soc.* Vol. XVI. For an interesting extension of the M. C. C to n-space and a proof thereby, vide F. P. White: *Proc. Camb. Phil. Soc.* Vol. 22 pp. 684-7.

In paper (iii) the inversive and projective view points are combined to prove Ramaswami Aiyar's result regarding the case in which the Clifford Circle of $2n+1$ lines is a straight line, and to generalise it to the case when the Clifford circle is one of constant radius. Lastly it is shown that the angles at which the circles cut at a point of multiple incidence are the same as those at which they cut at any other point of multiple incidence of the M. C. C. This is interesting as providing a generalisation of the well known theorem in elementary geometry that the angle between a tangent to a circle and a chord through the point of contact is equal to the angle in the opposite segment, which is the form taken by the result when $n=3$.

There is an analogous configuration in Laguerre geometry to the M. C. C. oriented lines and circles taking the place of points and circles and contact replacing incidence. Paper V proves the angle property of the M. C. C. in Möbius geometry obtained in the previous paper by elementary methods and proves also the analogous property which the result given above namely that in M. C. C. of Laguerre geometry the tangential segments marked out on any of the 2^{n-1} oriented lines of the configuration by all the n oriented circles which touch it are the same as on any other of the oriented lines.

PAPER VI

The range and power of Mathematics arises from her ability to isolate and deal with the essential features of a problem ignoring unessential details, and hence it arises that the same mathematical discipline is capable of dealing with several apparently diverse situations. It would be, therefore, more economical to study the central mathematical core in general terms rather than deal with particular realisations thereof. One such investigation is undertaken in paper (vi) where the basic discipline is the invariant theory of projective transformations in $[n]$ which leave a particular quadric Ω invariant. Particular realisations of this are:—

	points on Ω corresponding to
(1) Non-euclidean Geometry	points at infinity;
(2) Möbius geometry in $[n-1]$	points-spheres
(3) Turbine transformations in Lie's geometry in $[n-2]$	oriented spheres

(4) $n=5$ Projective line-geometry in [3] — straight lines (special linear complexes)

(5) Study's geometry of rigid displacements in [3] $n=7$ — Soma and pseudo-soma

In all these, the conjugacy of a pair of points in relation to Ω represents an important invariant relation. In paper (vi) I study certain linear transformations applied to sets of mutually conjugate elements which will conserve their conjugacy relations and give applications of these results to circle, sphere and line-geometries. Among others, certain interesting generalisations of the Miquel-Clifford and Hart systems are obtained.

The remaining papers deal with

9. Turbine Geometry.

The Turbine concept has been introduced into sphere geometry with a view to people the empty spaces in the representative $[n+2]$ where, so far, only points on Ω have been interpreted (as oriented spheres). The turbine is obtained by associating a non-oriented sphere C or radius R with an oriented sphere c of radius r in concentric position, and has thus an additional degree of freedom over the oriented sphere. It may be defined as an aggregate of oriented surface elements whose point locus is C and whose plane envelope is c. In fact we take every surface element † (oriented) of c and slide it in all directions in the tangent plane through the same distance t, the totality of surface elements thus obtained being the turbine. This is the inner specification (c, t) of the turbine through its inner sphere c and tangential segment t. The points of the elements lie on a sphere C of radius R given by $R^2=r^2+t^2$ and since R is thereby seen to be capable of a double sign, we regard C as oriented both ways which is my definition of "non-oriented". The elements cut C at a constant angle α where $\cos\alpha=r/R$. Now two primes cut at an angle which is determinate mod. 2π when they are both oriented and determinate only mod. π when one or both are non-oriented. Since the elements of C are oriented both ways the angle α is determinate mod. π and may be replaced by $\alpha+\pi$. In fact since $\cos\alpha=r/R$, we should take α if R has one sign and $\alpha+\pi$ if R takes the opposite sign, so that R $\cos\alpha$ which enters into the outer

† By a surface element is meant in hyperspace the association of a point and a prime incident therewith.

co-ordinates of a turbine (vide paper vii p. 99) is quite determinate. We thus have an alternative "outer specification" of the turbine through (C, α) in which each of the 2 possible signs for R is associated with a particular value out of α or $\alpha+\pi$. The significance of the non-oriented character of C lies precisely in this freedom to orient C either way, adjusting α accordingly. These considerations must lead us to expect in our formulae r and t^2 (since the sliding is done in all directions) or R^2 and $\cos^2 \alpha$.

When the inner sphere c is fixed and t allowed to vary we have an "inner pencil of turbines" which correspond to a line in $[n+2]$ through the point l, the invariant point of Laguerre geometry. If, on the other hand, C is fixed and α allowed to vary we have an "outer pencil" represented by a line through m, the invariant point of Mobius geometry, the 2 intersections of the line with Ω corresponding to C oriented different ways. In imbedding the concepts and results of Lie geometry and its sub-geometries into turbine geometry, the oriented sphere corresponds to the turbine whose inner and outer spheres coincide. Laguerre and Möbius geometries are specialised from Lie geometry by the fact that they carry inner pencils into inner pencils and outer pencils of turbines into outer pencils respectively. Euclidean geometry which is the intersection of the Laguerre and Möbius geometries is thus the sub-geometry of Lie which carries every outer pencil into an outer pencil and every inner pencil into an inner pencil. The non-oriented sphere of Mobius geometry may be imbedded into turbine geometry by making it correspond to a turbine whose inner sphere is of zero radius. The contact of two turbines is so defined that it corresponds to points whose join touches Ω (vide. postscript to paper vii)

The turbine concept was introduced in a tentative way in paper (vi) by means of "weighted spheres" and without a full idea of their geometric significance. The word itself is due to Kasner who defines it in two dimensions as the configuration obtained by turning the line elements of an oriented circle *the same way round* through the same angle. This has no analogue in higher dimensions and is called by me a "semi-turbine". My 2-dimensional turbine is obtained by rotating the line elements in *both senses* through the same angle and has thus 2 line elements through every point. The main features of the Kasner-turbine and the turbine in 3 dimensions are indicated in Blaschke's, *Vorlesungen uber Differentialgeometrie* Bd. III (1929) S. 51 and S. 231 as "linearer Komplex von Kugeln" but the subject is not pursued further nor are the concepts sharpened

for further use. In paper (vii) the foundations are laid for what promises to be an interesting extension of circle and sphere geometries in which the conformal and equilong patterns will intermingle with each other.

The first superstructure to be erected on these foundations is the creation of a pencil of geometries G_k bridging the span between Möbius and Laguerre geometries. The group which defines the geometry G_k is the group of collineations which carry Ω into itself and leave invariant a definite point K on the line *ml* joining the invariant points *m* and *l* of the Möbius and Laguerre geometries; or geometrically, the group of turbine transformations which carry oriented spheres into oriented spheres and turbines of inner radius k into others of inner radius k. By associating with each non-oriented sphere an inner oriented sphere of radius k, a G_k-geometry of non-oriented spheres is worked out which is the analogue of Möbius geometry. Spheres of radius k here play the same role as points in Mobius geometry, and there is a G_k inversion corresponding to Möbius inversion. The incidence of a point and a sphere is now replaced by the "bisection" of a k-sphere by a sphere of radius other than k while corresponding to the angle in Möbius geometry and the tangential segment in Laguerre geometry we have a basic invariant which partakes of the character of both. The Miquel-Clifford and Hart configurations are then generalised by transplantation into G_k-geometry. All these matters are discussed in (viii).

Since turbine geometry is essentially a geometry of suface elements, it seemed desirable to study the inversive plane as an aggregate of line elements—a study undertaken in paper (ix). The non-oriented line elements in the inversive plane are represented homeomorphically on a V_3^4 in a projective [5] which is the join of 2 conics whose planes are skew to each other. The inversive group induces in [5] projective transformations which have V_3^4 as an invariant variety. The structure of V_3^4 is studied and the loci thereon examined. Thus corresponding to line elements whose points lie on a circle, we have on V_3^4 a rational normal ruled surface (V_2^4) and on this we have "leitkurven" of lowest order 2 which correspond to turbines with the given circle for outer circle. "Leitkurven" of orders 3 and 4 are also interpreted. By projection from one of its generating lines on to a [3], we get a representation of line elements of the inversive plane by points in a [3]. The group induced in this [3] by the inversive group of the plane is a cremona cubocubic group some of whose features are studied. The configurations in [3]

corresponding to the line elements of a Hart System and of the Miquel Clifford Configuration are obtained. It is interesting to note in this connection that if a fixed point is kept invariant on the inversive plane (say the point at infinity) thus making the geometry euclidean, the group in [3] becomes the projective group which keeps invariant a line in [3], two points on it and two planes through it, a group whose geometry which has been studied both in its algebraic and differential aspects by Blaschke in his recent book *Ebene Kinematik* (Hamburger Math. Einzelschriften, Heft 25, 1938) under the name "Quasielliptische Geometrie". Quasielliptic space is no other than the parameter-space of the group of movements in the Euclidean plane and its group is the direct product of the group of movements with itself.

The last part of paper (ix) indicates some of the features of the analogue of line element geometry in the geometry G_k. It is found to be the geometry of "rods" (point pairs at a constant distance $2k$) under the group of G_k inversions.

In the last paper (x), I have studied the topology of the totality of line elements, both oriented and non-oriented in the inversive plane. The connection between the group of a geometry and the topology of the operand-manifold arises out of the need for a suitable closure of the latter so as to secure that each transformation of the geometry shall effect a (1,1) correspondence of the elements with themselves. It is shown that the oriented line elements of the inversive plane are topologically equivalent to the points of a projective 3-space,—a known result, since the manifold is homeomorphic with the oriented line elements of a 2-sphere. When the elements are not oriented the Poincare group is a cyclic group of order 4 and the variety is homeomorphic with a lenticular space in which points on each quadrant of the circular edge are considered identical with those obtained by a rotation about the axis through a right angle. I have not seen a specific statement of this result, though it almost follows by piecing together the known results in the topology of 3-dimensional manifolds.

Paper (x) concludes the thesis. Where known results have been used they are acknowledged in the body of the paper where they are quoted. All the other results are believed to be new.

Before concluding I may perhaps be allowed to indicate the directions in which further work may be carried on in which the ideas and results of turbine geometry will prove a valuable tool.

10. Generalisations and extensions of the Hart Configuration

The Hart Configuration consists of two tetrads of circles of which each circle of one tetrad touches all the circles of the other tetrad. This configuration belongs to Möbius geometry and has been the subject of elaborate study by many writers, particularly Coolidge (Treatise on the circle and the sphere), Study (*Mathematische Annalen* Bd. 49) and Baker (Principles of Geometry Vol. IV). That it will still repay study may be granted if it is pointed out that there are newly discovered facts such as the result by Dr. Vaidyanathaswamy that all the 8 circles of a Hart system belong to a cubic one-parameter family of circles which do not fit into the existing scheme, that nothing seems to be known about the existence of similar configurations in higher space, and that even in two dimensions the general features of two tetrads of circles which cut each other at the same angle do not appear to have been studied. Again, there should be analogous configurations in Laguerre geometry in which the length of the tangential segments between spheres one from each system is the same. The study of the Hart Configuration by means of oriented circles and spheres should prove somewhat easier since the high degree of irrationality involved in the Problem of Appolonius (as evidenced by the generous distribution of the $\sqrt{}$ sign in almost every page of Study's paper) gets reduced when oriented circles are used. Thus, touching 3 oriented circles there are only 2 oriented circles instead of 8 as in the case of non-oriented circles.

The existence of Hart pentads of spheres in [3] seems to follow from Schlafli's double-six of lines. For, the double-six corresponds on Plücker's Quadric in [5] to an in-circum simplex $A_1 \ldots A_6$ whose faces touch the quadric at say $\alpha_1 \ldots \alpha_6$. In other words we have two sets of points A_r, α_r $(r = 1, \ldots 6)$ such that $A_r \alpha_s$ is a generator if $r \neq s$ and not a generator when $r = s$. If we identify* Plucker's quadric with the Lie quadric Ω in [5] which images the oriented spheres in [3], we have 2 sets of 6 spheres of which each has proper contact with all the spheres of the other set except the corresponding member. Suppose now, that we can find a point O not on Ω such that OA_1, $OA_2 \ldots OA_5$ meet the opposite primes in points lying on Ω. Then OA_1 meets a generator through α_1, OA_2 meets a generator through α_2, and so on up to OA_5. We now subject the whole figure

* They do not have the same signature, so that the correspondence is not real→real.

to a collineation which carries Ω into itself and O into the point m which is the invariant point of Möbius geometry (vide paper vii p. 100), a point which, it will be remembered, does not lie on Ω. Denote the points after transformation by the same letters as before. The point sets $A_1 \ldots A_5$ and $\alpha_1 \ldots \alpha_5$ correspond to two pentads of spheres such that each of one set touches properly all the non-corresponding members of the other set and has improper contact with the corresponding member; for sphere A_1 with its orientation reversed is represented by the intersection of mA_1 with Ω and since this lies on a generator through α_1 it follows that A_1 reversed touches α_1. If the orientations are ignored we have a double-pentad of Hart spheres in Möbius Geometry.

Everything depends therefore, on the existence of a point O which will project $A_1 \ldots A_5$ into points on the corresponding section of Ω by their opposite faces; in other words, O is an intersection of the 5 cones with vertices at $A_1 \ldots A_5$ respectively and standing on the sections of Ω by their opposite primes. Now the first of these has $A_1 \alpha_1$ for a singular line and passes through $A_2 \ldots A_6$ and also through $\alpha_2 \ldots \alpha_6$. Thus A_1 and α_1 count each as 2 intersections of the 5 cones. For the second cone $A_2 \alpha_2$ is a double line and it is thus seen that out of the 32 intersections of the 5 cones in [5], $A_1 \ldots A_5$ $\alpha_1 \ldots \alpha_5$ count each for 2 intersections and A_6 and α_6 as single intersections, thus absorbing 22 intersections. There are therefore 10 more intersections, any one of which can be taken to be the point O. This would seem to establish the existence of Hart pentads of spheres in [3].

Similar considerations hold good in space of dimensions $n+2 >$ or $= 5$. Hence the existence of proper † in-circum simplexes in a quadric in 3 or higher dimensions will lead to the existence of Hart systems of spheres.

We shall next take up for consideration the

11. Spheres in $[n]$ which cut $n+2$ given spheres at the same angle.

We represent the given system of $n+2$ oriented spheres by points $A_1 \ldots A_{n+2}$ on the Lie quadric Ω in $[n+2]$.

These points lie on a prime whose pole with respect to Ω represents a turbine which is conjugate to all the given spheres. Hence

† *ie.* Simplexes for which the 2 sets A_r and α_r do not coalesce as happens for example in the case of a simplex referred to which Ω is $x_1 x_2 + x_3 x_4 + x_5 x_6 = 0$

its outer sphere cuts all the given spheres as the same angle while its inner sphere has with each of the given spheres the same tangential segment. If we reverse the orientation of one or more of the given spheres A_r we shall obtain other turbines conjugate to the spheres. Since reversing the orientation of all the spheres will give merely the turbine itself reversed, we see that if the given spheres are non-oriented, there are 2^{n+1} spheres each of which cuts them all at the same angle and also 2^{n+1} spheres concentric with the former which have each the same tangential segment with them (direct in some cases transverse in the others).

If we wish to study one of these sets, say the former, we may use cosine coordinates. Here the "coordinates" of a sphere are the cosines of the angles at which it cuts the given oriented spheres A_r. These cosine coordinates x_r satisfy a quadratic identity (vide vii p. 106).

$$x_1^2 \sec^2 \omega_r + \ldots + 2\gamma_{12} x_1 x_2 \sec \omega_1 \sec \omega_2 + \ldots = 1 \quad \ldots \quad (11\cdot1)$$

or briefly

$$f(x_1, \ldots x_{n+2}) \equiv \sum a_{rs} x_r x_s = 1 \quad \ldots \quad (11\cdot2)$$

where

$$a_{rr} = \sec^2\omega_r, \; a_{rs} = \gamma_{rs} \sec \omega_r \sec \omega_s \quad \ldots \quad (11\cdot3)$$

and ω_r is the angle between A_r and the sphere A_r'' orthogonal to all the spheres except A_r and γ_{rs} is the cosine of the angle between A_r'' and A_s''. If the orientation of the sphere be reversed all the cosine coordinates change their sign, and the ratio of the coordinates remains unaltered. Hence cosine coordinates may be considered as the homogeneous coordinates of the non-oriented sphere in a representative $[n+1]$. Since for a point sphere the cosine coordinates are infinite, the Mobius quadric in this space has the equation $f(x_1 \ldots x_{n+2}) = 0$.

The equiangular spheres of the given $n+2$ spheres have cosine coordinates which are equal. Hence they may be taken as 1, $\pm$ 1, $\pm$ 1 etc. Hence there are 2^{n+1} spheres of this type which separate into two distinct systems (one having an odd number and the other an even number of negative signs) when n is even and belong to one system when n is odd. The two systems have the property that their respective points are in $(n+2)$-ple perspective with the vertices of the reference simplex as centres of perspective; or to put it geometrically, the coaxal system formed by A_r'' and any member of one system contains also a member of the other system. Even if n is odd the coaxal system formed by any A_r'' and any equiangular circle contains another member.

The angle α at which a particular member say $-1, 1 \ldots 1$ cuts the given spheres is given by (11·1) with $x_1 = -\cos\alpha$, $x_2 = \cos\alpha$ etc.

$$f(-1, 1, \ldots 1) = \sec^2\alpha \qquad \ldots \quad (11\cdot4)$$

Hence the 2^{n+1} equiangular spheres will cut at different angles α unless the given spheres are orthogonal so that f consists of square terms only, i.e. $a_{rs} = 0$ $r \neq s$. In this case all the 2^{n+1} spheres cut the given spheres at the same angle α given by

$$\sec^2\alpha = \sum \sec^2\omega_r = n+2.$$

When $n=2$, this gives 60° as the angle of intersection. In this case we have 3 tetrads of orthogonal circles of which each circle of one system cuts all the circles of the other system at 60°.*

Coming again to the general case, the points $1, \pm 1, \pm 1 \ldots$ representing the equiangular spheres of the set A may be specified as the common points of quadrics in $[n+1]$ self polar with respect to the polar set A″ (representing the orthogonal circles) and outpolar to the Absolute $f(x_1 \ldots x_{n+2}) = 0$ representing point spheres. Of these let us select $n+2$ spheres, namely

$$\begin{aligned} B_1 &= (-1, 1, 1, \ldots \ldots 1) \\ B_2 &= (1, -1, 1, \ldots \ldots 1) \text{ etc.} \\ B_{n+2} &= (1, 1, 1, \ldots \ldots -1) \end{aligned}$$

If every one of these should cut every sphere A at the same angle α, we should have

$$\sec^2\alpha = f(-1, 1, 1, \ldots 1) = f(1, -1, \ldots 1) = f(1, 1, \ldots -1) \text{ etc.} \qquad \ldots \quad (11\cdot5)$$

Let us write

$$\begin{aligned} s_1 &= a_{12} + a_{13} + \cdots \cdots a_{1, n+2} \\ s_2 &= a_{21} + a_{23} + \cdots \cdots a_{2, n+2} \text{ etc.} \end{aligned}$$

where $a_{rs} = a_{sr}$ are the coefficients in (11·2). The equations (11·5) lead to

$$\sec^2\alpha = f - 4s_1 = f - 4s_2 = \cdots = f - 4s_{n+2} \qquad \ldots \quad (11\cdot6)$$

where $f = f(1, 1, \ldots 1)$. Hence it follows that

$$s_1 = s_2 = \cdots\cdots = s_{n+2} = \text{say } s$$

so that $f = a_{11} + a_{22} \cdots + a_{n+2, n+2} + (n+2)\ s$ which together with (11·6) leads to

* Vide Vaidyanathaswamy: On four mutually orthogonal circles, *Proc. of the Camb. Phil. Soc.* vol. 24 part 4.

$$\sec^2 \alpha = a_{11} + a_{22} + \cdots + a_{n+3,\, n+3} + (n-2)\, s$$

It is thus seen that 2-dimensional space is peculiar in that s removes itself, leading to

$$\sec^2 \alpha = \sec^2 \omega_1 + \sec^2 \omega_2 + \sec^2 \omega_3 + \sec^2 \omega_4 \qquad \ldots \quad (11\cdot7)$$

The conditions $s_1 = s_2 = s_3 = s_4$ are easily seen to be equivalent to $a_{12} = a_{34}$, $a_{13} = a_{24}$, $a_{14} = a_{23}$ or from (11·3),

$$\frac{\gamma_{12}}{\cos\omega_1 \cos\omega_2} = \frac{\gamma_{34}}{\cos\omega_3 \cos\omega_4}\text{; and 2 similar relations}\,\dagger \qquad \ldots \quad (11\cdot8)$$

The results for 2 dimensions may therefore be summarised as follows:

> Let A_r be a set of 4 circles and let their 8 equiangular circles be denoted by B_r'' and C_r'' (these corresponding to sets with an odd and with an even number of negative signs). Let A″, B, C denote the tetrads orthogonal to A, B″ and C″, so that we have the scheme
>
> A B C
>
> A″ B″ C″
>
> All quadrics through B″, C″ are outpolar to the absolute $f = 0$. If the circles B″ and A cut at equal angles, then all quadrics through A, C will also be outpolar to the absolute, and the condition for this to happen is that the set A should satisfy (11·8). The common angle of intersection α is then given by (11·7). The additional condition for a Hart-tetrad is given by putting $\alpha = 0$ in (11·7).

These results are scrappy and somewhat unrelated to the main body of known work in the field, which goes to show that the investigation has not penetrated deep enough yet.

† The tetrad A_r has given orientations and the orthogonal tetrad A_r'' may be oriented any way we please so as to definitize the various cosines which occur. If the orientation of any one—say A_1'' be reversed, γ_{1r} and $\cos\omega_r$ both change their signs leaving the equations (11·8) unaffected.

ON CORRELATION CONSTANTS IN MINGLED RECORDS

B. SITARAMAN, B.Sc. (Hon.)

Whenever a number of statistical records are mingled together, it is desirable to find out whether the single statistical constant derived from the mingled record represents or not the general tendency exhibited by similar constants, derived from the different records separately. S. Subramaniam † has discussed the case of the correlation coefficient and has shown that cases may arise where two characteristics are positively correlated in each record separately while the mingled record exhibits a negative correlation. The object of the present paper is to discuss the behaviour of the regression, coefficient as well as the intra-class correlation coefficient.

Let there be k records, and let x and y be the variables. Taking the sth record, let N_s, M_{sx}, M_{sy}, σ_{sx}, σ_{sy}, γ_s, b_s $[=\gamma_s \sigma_{sy}/\sigma_{sx}]$ be the number of observations, the means, the standard deviations, the correlation coefficient, and the regression coefficient respectively; and let the constants for the whole set after mingling be N, M_x, M_y, σ_x, σ_y, γ b $[=\gamma\sigma_y/\sigma_x]$ respectively. We then have the following relations.—

$$N = \sum_{s=1}^{k} N_s.$$

$$M_x = \frac{\sum N_s M_{sx}}{N}; \quad M_y = \frac{\sum N_s M_{sy}}{N}$$

$$N\sigma_x^2 = \sum N_s \sigma_{sx}^2 + \sum N_s (M_{sx} - M_x)^2$$

$$N\sigma_y^2 = \sum N_s \sigma_{sy}^2 + \sum N_s (M_{sy} - M_y)^2$$

$$\gamma = \frac{\sum N_s \gamma_s \sigma_{sx} \sigma_{sy} + \sum N_s (M_{sx} - M_x)(M_{sy} - M_y)}{N\sigma_x \sigma_y} \quad \ldots \quad (1)$$

Now $b = \gamma\, \sigma_y/\sigma_x$

$$= \frac{\sum N_s \gamma_s \sigma_{sx} \sigma_{sy} + \sum N_s (M_{sx} - M_x)(M_{sy} - M_y)}{\sum N_s \sigma_{sx}^2 + \sum N_s (M_{sx} - M_x)^2}$$

$$= \frac{\sum N_s b_s \sigma_{sx}^2 + \sum N_s (M_{sx} - M_x)(M_{sy} - M_y)}{\sum N_s \sigma_{sx}^2 + \sum N_s (M_{sx} - M_x)^2}$$

---

† S. Subramaniam *The Mathematics Student* Vol. V., No. 3.

Let R, B and S_x be the statistical constants corresponding to the sample formed by the pairs of means $(M_{1x}, M_{1y}) \ldots\ldots (M_{kx}, M_{ky})$ with the frequencies $N_1, N_2, \ldots, N_k$ respectively. Then it follows that

$$b = \frac{\sum N_s b_s \sigma_{sx}^2 + NBS_x^2}{\sum N_s \sigma_{sx}^2 + NS_x^2} \quad \ldots \quad (2)$$

Therefore, if we take k records each involving two variables x and y,

the regression coefficient of y upon x of the mingled record is the weighted arithmetic mean of the $(k+1)$ regression coefficients of the k records and the set of means, the weights being the corresponding sums of squares of x in the $(k+1)$ sets.

In the particular case when $b_1 = b_2 = \cdots b_k = \lambda$, b is also equal to λ, only when

$$\lambda = \frac{\sum N_s \lambda \sigma_{sx}^2 + NBS_x^2}{\sum N_s \sigma_{sx}^2 + NS_x^2}$$

$$= \frac{\lambda \sum N_s \sigma_{sx}^2 + NBS_x^2}{\sum N_s \sigma_{sx}^2 + NS_x^2}$$

i.e. $\lambda \sum N_s \sigma_{sx}^2 + \lambda NS_x^2 = \lambda \sum N_s \sigma_{sx}^2 + NBS_x^2$

i.e. $N(\lambda - B) S_x^2 = 0 \quad \ldots \quad (3)$

$\therefore$ Either $\lambda = B$ or $S^2_x = 0$, since $N \neq 0$.

Hence,

if the regression coefficient of y upon x is the same for each of k records separately, it is unaffected by mingling if either

(1) *it is equal to the regression coefficient of the sample formed by the means of the k records, or*

(2) *the means of x in the several records are all equal.*

We shall now consider the case of the intra-class correlation *i.e.* correlation between two variables of identical character. Here, if (x_{st}, y_{st}) is a pair of values, (y_{st}, x_{st}) is another pair, so that the mean and the standard deviation are the same for each variable; and the intra-class correlation coefficient is given by

$$\gamma_s = \frac{\sum_{t=1}^{N_s} (x_{st} - M_s)(y_{st} - M_s)}{N_s \sigma_s^2}$$

Now, the correlation coefficient after mingling is given by

$$\gamma = \frac{\sum N_s \gamma_s \sigma_{sx} \sigma_{sy} + \sum N_s (M_{sx} - M_x)(M_{sy} - M_y)}{N\sigma_x \sigma_y}$$

But here $M_{sx} = M_{sy}$ $(= M_s)$ and $\sigma_{sx} = \sigma_{sy} = \sigma_s$; so we have on substitution

$$\gamma = \frac{\sum N_s \gamma_s \sigma_s^2 + NS^2}{\sum N_s \sigma_s^2 + NS^2} \qquad \dots (4)$$

Since the correlation coefficient is always less than or equal to unity,

$$|\sum N_s \gamma_s \sigma_s^2| \leqslant \sum N_s \sigma_s^2 \qquad \dots (5)$$

Hence, by the relation $\frac{p}{q} \leqslant \frac{p+\mu}{q+\mu}$ where μ and q are positive and $|p| \leqslant q$, we have from relations (4) and (5)

$$\frac{\sum N_s \gamma_s \sigma_s^2}{\sum N_s \sigma_s^2} \leqslant \gamma \leqslant 1 \qquad \dots (6)$$

i.e. the intra-class correlation coefficient after mingling lies between unity and the weighted mean of the coefficients of the several records, the weights being the corresponding 'sums of squares'. It attains the value unity when the correlation coefficient of each record is also unity.

Further, since $NS^2 \geqslant 0$, the sign of γ depends upon $\sum N_s \gamma \sigma_s^2$. *Therefore if positive intra-class correlation is exhibited in each of the k records, then the mingled record must also exhibit a positive intra-class correlation.* On the other hand, if intra-class correlation is negative in each record, the mingled record will also exhibit a negative correlation only when $NS^2 < |\sum N_s \gamma_s \sigma_s^2|$. Otherwise we have the case of a mingled record showing a positive intra-class correlation, when each record exhibits a negative intra-class correlation separately.

AN APPROXIMATION CONNECTED WITH EXP. x

DR. S. CHOWLA AND F. C. AULUCK, *Lahore.*

1. Ramanujan proved that $S(x) \sim \frac{1}{2} e^x$ where

$$S(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^r}{r!};$$

but the proof as given by him in the *Collected Papers* (p. 324) is not rigorous. In this note we give a rigorous proof of the above theorem. It may be remarked that the whole contribution of $S(x)$ is from a *small* interval of r in the neighbourhood* of x. In fact

$$S(mx) = o\,(e^x) \quad \text{if } m < 1 \text{ and } S(mx) \sim e^x \text{ if } m > 1.$$

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* Hardy and Wright. *An Introduction to the Theory of Numbers* (1938) p. 360.
Polya and Szego, *Aufgaben und Lehrsatze aus der Analysis*, Erster Band (1925) p. 80.

Ramanujan's problem was to prove that y lies between $\frac{1}{3}$ and $\frac{1}{2}$ where y satisfies the equation

$$\frac{1}{2} e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^x}{x!} y.$$

A 'partial proof' was given by him in which he proved that $y = 1/2$, when $x = 0$ and $y \to 1/3$ when $x \to \infty$. Later proofs were supplied by Watson[1] and Szego[2]. Watson gave some reasons for believing that

$$y = \frac{1}{3} + \frac{1}{135\,(x+k)}$$

where k lies between $\frac{8}{45}$ and $\frac{2}{21}$, but no complete proof has been given for the above statement.

2.
$$S(x) = \frac{x^x}{x!} \int_0^\infty e^{-z} \left(1 + \frac{z}{x}\right)^x dz$$

$$= \frac{x^x}{x!} \left\{ \int_0^{x\phi} + \int_{x\phi}^{2x/3} + \int_{2x/3}^{\infty} \right\}$$

$$= \frac{x^x}{x!} \left\{ I_1 + I_2 + I_3 \right\}$$

where $\frac{1}{2} < \varphi < \frac{2}{3}$.

In I_3, $z \geqslant \frac{2x}{3}$, $1 + \frac{z}{x} \leqslant \frac{5}{2}\frac{z}{x}$; hence

$$0 < I_3 < \int_0^\infty e^{-z} \left(\frac{5}{2}\right)^x \left(\frac{z}{x}\right)^x dz = \frac{x!}{x^x} \left(\frac{5}{2}\right)^x$$

In I_2, the integrand is

$$\exp\left\{ -z + x \log\left(1 + \frac{z}{x}\right) \right\}$$

$$= \exp\left\{ -\frac{z^2}{2x}\left(1 + \frac{\theta z}{x}\right)^{-2} \right\} \quad (0 < \theta < 1)$$

$$\leqslant \exp\left\{ -\frac{1}{2} x^{2\phi - 1} \left(1 + \frac{2}{3}\right)^{-2} \right\}$$

$$= \exp\left(-\frac{9}{50} x^{2\phi - 1} \right)$$

Hence $I_2 \to 0$ as $x \to \infty$ since $\phi > \frac{1}{2}$

(1) *Proc. of London Math. Soc.* Ser. 2. Vol 29. Part 4 (1928) 293—308.

(2) *Journal of the London Math. Soc.* 3 (1928) 225—232

Finally $I_1 = \int_0^{x^\phi} \exp\left\{-\frac{z^2}{2x} + \frac{\theta z^3}{x^2}\right\} dz. \quad (0 < \theta < 1)$

$$= \sqrt{2x}\int_0^G \exp\left\{-u^2 + \frac{\theta 2\sqrt{2}\, u^3}{\sqrt{x}}\right\} du$$

where $z = (\sqrt{2}\ \sqrt{x}\ u)$ and $G = x^{\phi - \frac{1}{2}}\sqrt{2}$.

$$\text{But } \exp\left\{\frac{2\sqrt{2}\ \theta u^3}{\sqrt{x}}\right\} = 1 + O\left(\frac{u^3}{\sqrt{x}}\right)$$
$$= 1 + O\left(x^{3\phi - 2}\right)$$

Since $3\varphi < 2$, it follows that

$$I_1\ \sqrt{2x} \rightarrow \int_0^\infty e^{-u^2}\, du = \frac{\sqrt{\pi}}{2}$$

$$\text{Hence } S(x)\ e^{-x} \backsim \frac{x^x}{x!}\ e^{-x}\ \frac{\sqrt{\pi}}{2}\ \sqrt{2x} \backsim \tfrac{1}{2},$$

since by Stirling's Theorem $x! \backsim \sqrt{2\pi x}\ (x/e)^x$.

3. $\frac{x^r}{r!}$ is maximum at $r = x$. Hence, if $0 < m < 1$.

$$S(mx) < mx\ \frac{x^{mx}}{\Gamma(mx)}$$
$$\backsim \frac{mx\ x^{mx}}{\sqrt{(2\pi mx)}}\ (mx/e)^{-mx}$$
$$\backsim \sqrt{\frac{mx}{2\pi}}\ (e/m)^{mx}$$

As $(e/m)^m < e^x$ if $0 < m < 1$,
$S(mx)\ e^{-x} \rightarrow 0$

Finally we notice that

$$\sum_{r=x+1}^{\infty} \frac{x^r}{r!} \backsim \tfrac{1}{2} e^x$$

$$\text{or } \frac{x}{x+1} + \frac{x^2}{(x+1)(x+2)} + \cdots \backsim \sqrt{(\tfrac{1}{2}\pi x)}$$

If $m > 1$ and assuming mx to be integral

$$\frac{x^{x(m-1)+1}}{(x+1)(x+2)\cdots(mx+1)} + \frac{x^{x(m-1)+2}}{(x+1)(x+2)\cdots(mx+2)} + \cdots$$
$$= \frac{x^{x(m-1)+1}}{(x+1)(x+2)\cdots(mx+1)}\left\{1 + \frac{x}{mx+2} + \frac{x^2}{(mx+2)(mx+3)} + \cdots\right\}$$
$$< 1 + \frac{1}{m} + \frac{1}{m^2} + \cdots = \frac{m}{m-1}.$$

Hence $S(mx)\ e^{-x} - 1 \rightarrow 0$.

ON WEIRESTRASS APPROXIMATION THEOREM

Dr. S. Chowla and F. C. Auluck

1. It is known* that any function $f(x)$, continuous in an interval $a \leqslant x \leqslant b$ can be approximated uniformly through a polynomial $P(x)$; that is, the difference $|f(x) - P(x)|$ can be made less than ε which is independent of x. In the present note we prove that

$$n = O\left(\varepsilon^{-2/k} \log 1/\varepsilon\right)$$

where n is the degree of $P(x)$ and $f(x)$ satisfies Lipschitz condition $f(x+\theta) - f(x) = O(\theta^k)$, k being a positive number, independent of x. It follows that for all differentiable functions

$$n = O\left(\varepsilon^{-2} \log 1/\varepsilon\right)$$

It may be mentioned that most of the ordinary functions like $\sin x$, $\cos x$, e^x, require only $O\left(\dfrac{\log 1/\varepsilon}{\log \log 1/\varepsilon}\right)$ terms.

2. For the sake of simplicity, we take the interval (a, b) to lie wholly within the interval (0,1). Consider the integrals

$$J_n = \int_0^1 (1 - v^2)^n \, dv, \quad J_n^* = \int_\delta^1 (1 - v^2)^n \, dv$$

δ being a fixed number in the interval (0,1).

Since $\qquad J_n > \int_0^1 (1 - v)^n \, dv = \dfrac{1}{n+1}$

and $\qquad J_n^* < (1 - \delta^2)^n (1 - \delta) < (1 - \delta^2)^n$

therefore $\quad J_n^*/J_n < (n+1)(1 - \delta^2)^n \qquad \dots \quad (1)$

$$P_n(x) = \frac{\int_\alpha^\beta f(u)\,[1 - (u - x)^2]^n \, du}{\int_{-1}^{1} (1 - u^2)^n \, du} \quad (0 < \alpha < a < b < \beta < 1) \qquad \dots \quad (2)$$

is a polynomial in x of degree $2n$.

$$\text{The numerator} = \int_{\alpha - x}^{\beta - x} f(v + x)(1 - v^2)^n \, dv$$

$$= \int_{\alpha - x}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\beta - x}$$

$$= I_1 + I_2 + I_3$$

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* Courant and Hilbert, *Methoden der Mathematischen Physik I.* (1931) pp. 55—57.

If M is the maximum of $|f(x)|$ in $\alpha \leqslant x \leqslant \beta$

$$|I_1| < M \int_{-1}^{-\delta} (1-v^2)^n \, dv = M \, J_n^*$$

$$|I_3| < M \int_{\delta}^{1} (1-v^2)^n \, dv = M \, J_n^*$$

$$I_2 = f(x) \int_{-\delta}^{\delta} (1-v^2)^n \, dv + \int_{-\delta}^{\delta} [f(v+x) - f(x)] \, (1-v^2)^n \, dv$$

$$= 2f(x)(J_n - J^*_n) + \int_{-\delta}^{\delta} [f(v+x) - f(x)] \, (1-v^2)^n \, dv$$

Assume that $|f(v+x) - f(x)| < Av^k$ where A and k are positive and independent of v. So

$$|f(v+x) - f(x)| < \varepsilon/2$$

if $a \leqslant x \leqslant b$, $|v| \leqslant \delta < \left(\frac{\varepsilon}{2A}\right)^{1/k}$ ... (3)

Therefore $\left| \int_{-\delta}^{\delta} [f(v+x) - f(x)] \, (1-v^2)^n \, dv \right|$

$$< (\varepsilon/2) \int_{-\delta}^{1} (1-v^2)^n \, dv$$

$$< (\varepsilon/2) \int_{-1}^{1} (1-v^2)^n \, dv$$

$$= \varepsilon \, J_n$$

Hence $|f(x) - P_n(x)| < 2M \, J_n^*/J_n + \varepsilon/2$

$$< 2(n+1)(1-\delta^2)^n \, M + \varepsilon/2$$

$$< \varepsilon$$

if $(n+1)(1-\delta^2)^n < \varepsilon/4M$ ... (4)

or $n \log \frac{1}{1-\delta^2} - \log (n+1) > \log 4M/\varepsilon$.

and therefore $n = O\left(\frac{1}{\delta^2} \log 1/\varepsilon\right)$

$$= O\,(\varepsilon^{-2/k} \log 1/\varepsilon) \text{ by (3)}$$

3. In general the coefficients of the polynomial $P(x)$ are functions of n. For example consider the function $1/x$ $(1 \leqslant x \leqslant 2)$

Substituting $x = \frac{3}{2} + y$ $(-\frac{1}{2} < y < \frac{1}{2})$

we obtain $\frac{1}{x} = \frac{1}{\frac{3}{2}+y}$

$$= \frac{2}{3} \left\{ 1 - \frac{2y}{3} + \left(\frac{2y}{3}\right)^2 \cdot \cdot \cdot \cdot + (-1)^{n-1} \left(\frac{2y}{3}\right)^{n-1} \right\} + k_n$$

$$= P_n(x) + k_n$$

where $|k_n| = \frac{2}{3} \cdot \left(\frac{2y}{3}\right)^n \left(1 + \frac{2y}{3}\right)$

$$< \varepsilon \text{ if } n = O\,(\log 1/\varepsilon)$$

Obviously the coefficients of $P_n(x)$ are functions of n.

NOTES AND DISCUSSIONS

Heliacal rising of the Moon

The moon's visibility for the first time in the evening after new moon determines the beginning of the Hijri month. The problem is of some interest to this Observatory as the day of the heliacal rising of the moon fixes the commencement of the several months in the Hijri calendar. As this phenomenon is to a great extent dependent on the times of sunset and moonset, the duration of twilight, the state of the sky etc. it is difficult to lay down hard and fast rules for determining the day on which the moon can be seen in critical cases.

I may note in passing, that the problem will be of special interest, from the religious point of view, to persons belonging to the *Bodhayana Sutra* as the moon must be visible on the third day of their observing the *Amavasya.* According to the Hindus, if the first *Tithi* ends before the 25th *ghati* or in other words if the elongation of the moon be 12 degrees two hours before sunset, the appearance of the thin crescent is certain. If the *tithi* ends between the 25th and 35th *ghatis*, the appearance is doubtful and after the 35th *ghati* it is impossible.

According to another and a better method, the method of *Kalamsas* as it is called, the interval between the times of sunset and moonset is calculated; and if it exceeds 50 minutes, visibility is believed to be certain.

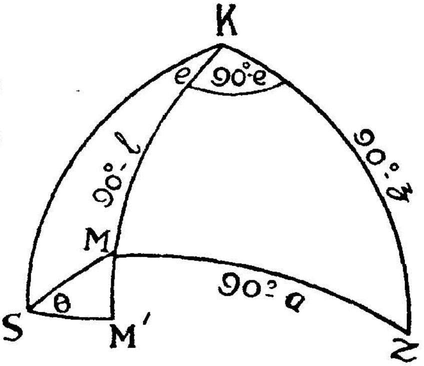
According to the late Mr. Ketkar, the visibility of the moon is certain if her altitude is greater than 12 degrees. He gives the following rule for determining the moon's visibility. If z denotes the zenith distance of the nonagesimal, e the elongation of the moon at sunset and l the latitude of the moon, visibility of the moon will be certain, doubtful or impossible according as $e \cos z + l \sin z >=$ or <12 degrees respectively. (z is treated as positive when south of the zenith).

The limit of 12 degrees is rather too much. Further the above expression for altitude is only a very rough approximation. The more rigorous expression for the altitude based on the zenith distance of the nonagesimal is derived as follows :—

Let Z denote the zenith

K the pole of the ecliptic

M the moon
S the sun at sunset
M′ the foot of the secondary to the ecliptic through M
a the altitude of the moon
l the latitude of the moon
z the zenith distance of the nonagesimal
SKM $= e$, the elongation of the moon
M S M′ $= \theta$
Then SKZ $= 90^\circ$
$\therefore$ MKZ $= 90^\circ - e$



From the Δ MKZ we have,

$\sin a = \sin l \sin z + \cos l \cos z \sin e$

$= \sin SM \sin\theta \sin z$
$+ \sin SM \cos\theta \cos z$
$= \sin SM \cos(z - \theta)$

$$a = \text{Sin}^{-1} [\sin SM \cos(z - \theta)].$$

where $\quad SM = \cos^{-1}(\cos l \cos e)$

and $\quad \theta = \tan^{-1}[\tan l \text{ cosec } e]$

If ϕ denotes the latitude of the place and ω the obliquity of the ecliptic, z lies between $\phi + \omega$ and $\phi - \omega$; also the maximum numerical value of θ is $23^\circ 25'$ taking $5^\circ 9'$ for the mean value of l and 12° for e.

a is minimum when SM is minimum and $z - \theta$ maximum (i.e.) z maximum and θ minimum. In Hyderabad, for example, the maximum value of z is $40^\circ 53'$ so that the minimum value of a can be as low as 5 or 6° though the moon's elongation may be 12° or 13°. This will occur in the most unfavourable case when the ecliptic is least inclined to the horizon, as at the time of the autumnal equinox and the moon has the maximum southern latitude. Though this may be of rare occurrence, it shows that the elongation criterion is not reliable for determining the visibility of the moon.

After examining a number of critical cases I have found that the altitude criterion is the safest for determining the visibility of the moon. If her altitude be greater than 11° we can safely predict the appearance of the crescent in our latitudes: when the altitude lies between 10° and 11° the appearance is doubtful and has to be decided by considering other factors such as the duration of twilight etc. When it is less than 10°, the appearance is impossible.

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A Note on Cauchy's Theorem

Cauchy's Theorem :—If $f(z)$ is holomorphic inside and continuous on the closed contour C, then $\int_C f(z)\, dz = 0$. From the given conditions we get,

(i) To a given ε corresponds an η such that $\left| \frac{f(z) - f(z')}{z - z'} - f'(z) \right| < \varepsilon$ if $|z - z'| < \eta$, η being independant of the positicns of the two points z and z' inside or on a closed contour γ lying inside C. (This is called Goursat's Lemma).

(ii) To a given ε' corresponds an η' such that $| f(z) - f(z')| < \varepsilon'$ if $|z - z'| < \eta'$, η' being independant of the positions z and z' inside or on C. (Uniform continuity).

(iii) $\int_\gamma f(z)\, dz + \int_\gamma z f'(z)\, dz = 0$, γ being entirely inside C.

To prove this last result, let δ be the greatest distance of the origin from C, and l be the length of the contour γ. Then

$$\int_\gamma f(z)\, dz = \lim \sum (z_i - z_{i-1})\, f(z_i)$$

$$= - \lim \sum z_i \left[f(z_{i+1}) - f(z_i) \right] = - \lim \sum z_i \left[f'(z_i) + \varepsilon_i \right](z_{i+1} - z_i)$$

$$\therefore \left| \int_\gamma f(z)\, dz + \int_\gamma z f'(z)\, dz \right| = \left| - \sum z_i\, \varepsilon_i\, (z_{i+1} - z_i) \right| < \delta \varepsilon l$$

since $|\varepsilon_i| < \varepsilon$ ultimately by (i)

Proof :—We take the origin to be any convenient point inside C. We assume that the curve is such that any vector from O, the origin meets the curve in only one point.

If P be any point on C divide OP into n equal parts at $P_1, P_2, \ldots P_{n-1}$. Then as P moves on C, P_r describes a similar curve C_r, $r = 1, 2 \ldots, n-1$. Take n greater than δ/η and $\delta\, \eta'$. Let L be the length of the curve C and M be the upper bound of $f(z)$ inside and on C.

If P is z_i then P_r is $z_i\ r\ n$

Also $f\left(z_i \frac{r+1}{n}\right) - f\left(z_i \frac{r}{n}\right) = \frac{z_i}{n}\left[f'\left(z_i \frac{r}{n}\right) + \varepsilon_i \right]$, $|\varepsilon_i| < \varepsilon$ by (i)

$$\therefore \int_{C_{r+1}} f(z)\, dz = \lim \sum \frac{r+1}{n} \left(z_i - z_{i-1} \right) f\left(z_i \frac{r+1}{n} \right)$$

$$= \lim \sum \frac{r+1}{n} \left(z_i - z_{i-1} \right) \left[f\left(z_i \frac{r}{n} \right) + \frac{z_i}{n} f'\left(z_i \frac{r}{n} \right) + \varepsilon_i \frac{z_i}{n} \right]$$

$$= \frac{r+1}{r} \int_{C_r} f(z)\, dz + \frac{r+1}{r^2} \int_{C_r} z f'\, dz + \sum \frac{r+1}{n}\, \varepsilon_i \frac{z_i}{n} \left(z_i - z_{i-1} \right)$$

$$\therefore \quad \int_{C_{r+1}} f dz - \int_{C_r} f dz = -\frac{1}{r^2}\int_{C_r} f dz + \frac{r+1}{n^2}\sum z_i\,\varepsilon_i\,(z_i - z_{i-1}) \text{ by (iii)}$$

$$\therefore \quad \left|\int_{C_{r+1}} - \int_{C_r}\right| \leqslant \frac{1}{r^2}\left|\int_{C_r} f dz\right| + \frac{r+1}{n^2}\left|\sum z_i\,\varepsilon_i\,(z_i - z_{i-1})\right|$$

$$\leqslant \frac{1}{r^2}\,M\left(\frac{Lr}{n}\right) + \frac{r+1}{n^2}\,\varepsilon\delta\,L \leqslant \frac{ML}{rn} + \frac{\varepsilon\delta L}{n}$$

$$\left|\int_{C_1} f dz\right| \leqslant \frac{ML}{n}$$

$$\left|\int_C - \int_{C_{n-1}}\right| = \left|\sum(z_i - z_{i-1})f(z_i) - \sum \frac{n-1}{n}(z_i - z_{i-1})\,f\left(z_i\,\frac{n-1}{n}\right)\right|$$

$$\leqslant \left|\sum(z_i - z_{i-1})\left[f(z_i) - f\left(z_i\,\frac{n-1}{n}\right)\right]\right| + \frac{1}{n}\left|\sum(z_i - z_{i-1})\,f\left(z_i\,\frac{n-1}{n}\right)\right|$$

$$\leqslant L\,\varepsilon' + \frac{ML}{n}$$

$$\therefore \quad \left|\int_C f dz\right| \leqslant \left|\int_C - \int_{C_{n-1}}\right| + \sum_{r=1}^{n-2}\left|\int_{C_{r+1}} - \int_{C_r}\right| + \left|\int_{C_1}\right|$$

$$\leqslant L\,\varepsilon' + \frac{ML}{n} + \frac{ML}{n}\left(\sum_{r=1}^{n-2}\frac{1}{r}\right) + \varepsilon\delta\,L + \frac{ML}{n}$$

Making $n \to \infty$ the left hand side is arbitrarily small. Now any region enclosed by a simple curve can be divided into a finite number of regions in each of which we can find a point such that any vector from the point meets its contour once only, and sum of the integrals round such contours vanishes.

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Lecture Notes

I—Reduction of Cylinders and Paraboloids

§1. The general equation of a cylinder (other than parabolic) or a paraboloid can be put in the form

$$(lx + my + nz)(l'x + m'y + n'z) + 2ux + 2vy + 2wz + d = 0 \quad \ldots \quad (1)$$

which may be re-written as

$$(lx + my + nz + \lambda)(l'x + m'y + n'z + \mu) = u'x + v'y + w'z + d'$$

where $u' = \lambda l' + \mu l - 2u$

$v' = \lambda m' + \mu m - 2v$

$w' = \lambda n' + \mu n - 2w$

$d' = \lambda\mu - d$, and $(l', m', n') \neq (l, m, n)$ $\quad \ldots \quad (2)$

(i) If $\begin{vmatrix} u & v & w \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0,$

find λ, μ so that $u' = v' = 0$ and therefore $w' = 0$.

Taking as two co-ordinate planes of reference ($X = 0$, $Y = 0$) the planes bisecting the angles between

$$lx + my + nz + \lambda = 0$$
$$l'x + m'y + n'z + \mu = 0$$

and for the third plane any plane perpendicular to these, the equation (1) reduces to the Standard form

$$X^2 \sum (l + kl')^2 - Y^2 \sum (l - kl')^2 = 4k \; (\lambda\mu - d) \quad \dots \quad (3)$$

where $k^2 = \sum l^2 / \sum l'^2$, and the equation generally represents a cylinder.

(ii) If $\begin{vmatrix} u & v & w \\ l & m & n \\ l' & m' & n' \end{vmatrix} \neq 0$, find λ, μ so that

$$u'l + v'm + w'n = 0 = u'l' + v'm' + w'n'$$

Then the plane $u'x + v'y + w'z + d' = 0$ will be perpendicular to $lx + my + nz + \lambda = 0$, and $l'x + m'y + n'z + \mu = 0$ and therefore to the planes bisecting the angles between these. With the bisecting planes as two co-ordinate planes and $u'x + v'y + w'z + d' = 0$ as the third co-ordinate plane, our original equation will reduce to the standard form

$$X^2 \sum (l + kl')^2 - Y^2 \sum (l - kl')^2 = 4kZ \; (\sum u'^2)^{\frac{1}{2}} \quad \dots \quad (4)$$

k having the same significance as in (3).

The above method is sufficient for numerical work, provided the terms of the second degree in (1) factorise rationally.

§2. If the equation of the conicoid be given by

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

we may express the coefficients in the reduced equations (3) and (4) in terms of those of the given equation.

In the case of the cylinder, the calculation may be made thus:

$$\sum (l + kl')^2 = 2k \left(\pm \sum ll' + \frac{\sum l^2}{k} \right) = 2k \; (\pm ll' + T) \text{ (say)} \quad \dots \quad (5)$$

If $f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$
$$\equiv (lx + my + nz)(l'x + m'y + n'z)$$

we have
$$\left.\begin{array}{l} a = ll', \quad b = mm', \quad c = nn' \\ 2f = mn' + m'n, \; 2g = nl' + n'l, \; 2h = lm' + l'm, \end{array}\right\} \quad \ldots \quad (6)$$

$$\begin{aligned} T^2 &= \sum l^2 \sum l'^2 \\ &= (\sum ll')^2 + \sum (mn' - m'n)^2 \\ &= (\sum ll')^2 + \sum (mn' + m'n)^2 - 4 \sum mn\, m'n' \\ &= (\sum a)^2 + 4 \sum f^2 - 4 \sum ab \quad \text{by (6)} \end{aligned}$$

i.e. $T = (a^2 + b^2 + c^2 - 2ab - 2bc - 2ca + 4f^2 + 4g^2 + 4h^2)^{\frac{1}{2}}$ ... (7)

λ, μ being given by $u' = v' = w' = 0$,

we have $\lambda\mu = \dfrac{4(um - vl)(um' - vl')}{-(lm' - l'm)^2}$ = two similar expressions

$$= \frac{4 \sum (bu^2 + av^2 - 2huv)}{-\sum (lm' - l'm)^2}$$

$$= \frac{\sum a \sum u^2 - f(u, v, w)}{\sum ab - \sum f^2} \quad \ldots \quad (8)$$

The reduced equation is
$$(a + b + c + T)\, x^2 + (a + b + c - T)\, y^2 = 2(\lambda\mu - d). \quad \ldots \quad (9)$$

Illustration :

$2y^2 - 2yz + 2zx - 2xy - x - 2y + 3z - 2 = 0$

i.e. $(2y - 2z)(y - x) - x - 2y + 3z - 2 = 0$

Here $\begin{vmatrix} -1 & -2 & 3 \\ 0 & 2 & -2 \\ -1 & 1 & 0 \end{vmatrix} = 0.$ Therefore the equation represents a cylinder.

$k^2 = \frac{8}{2} = 4$; we may take $k = 2$.

λ, μ are given by

$$-\lambda + \mu \cdot 0 = -1$$
$$\lambda + 2\mu = -2$$

Hence $\lambda = 1, \; \mu = -\frac{3}{2}$

The reduced equation is, therefore, given by

$$24X^2 - 8Y^2 = 8 \times \tfrac{1}{2} = 4$$
i.e. $$6X^2 - 2Y^2 = 1$$

Or, from (9) directly, since $a + b + c = 2$, $T = 4$, $\lambda\mu - d = 1 \cdot 2$ we get the same equation.

In the case of paraboloids, (5), (6), and (7) hold good as before and we have to calculate only $\sum u'^2$.

Since $u'l+v'm+w'n=0$, $u'l'+v'm'+w'n'=0$,

we have

$$\frac{u'}{mn'-m'n}=\frac{v'}{nl'-n'l}=\frac{w'}{lm'-l'm}=\frac{(u'^2+v'^2+w'^2)^{\frac{1}{2}}}{\{\sum(mn'-m'n)^2\}^{\frac{1}{2}}}$$

$$=\frac{\sum u'(mn'-m'n)}{\sum(mn'-m'n)^2}$$

$$\therefore\ (\sum u'^2)^{\frac{1}{2}}=\{\sum(mn'-m'n)^2\}^{-\frac{1}{2}}\sum u'(mn'-m'n)$$

But from (2), $\sum u'(mn'-m'n)=-2\sum u(mn'-m'n)\neq 0$, by our original assumption.

Hence $\sum u'^2$ does not vanish and the equation (4) always represents a paraboloid.

From (6), $(mn'-m'n)$, $(nl'-n'l)$, $(lm'-l'm)$ are respectively proportional to A, H, G which are the cofactors of a, h, g respectively in the vanishing determinant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Hence $$(\sum u'^2)^{\frac{1}{2}}=-\frac{2(uA+vH+wG)}{(A^2+H^2+G^2)^{\frac{1}{2}}} \qquad \ldots (10)$$

The reduced equation of a paraboloid takes the form

$$(a+b+c+T)X^2+(a+b+c-T)Y^2=-\frac{4k(uA+vH+wG)}{(A^2+H^2+G^2)^{\frac{1}{2}}} \qquad \ldots (11)$$

Cor: $uA+vH+wG\neq 0$ for a paraboloid.

Illustration:

$$4y^2+4z^2+4yz-2x-14y-22z+33=0$$

Here the terms of the second degree do not factorise rationally and therefore the method of § 1 is not applicable easily.

We fall back on (11), using the values of T and A, H, and G.

We find $A=12$, $H=0$, $G=0$, $u=-1$ and $uA+vH+wG=-12$, $T=4$.

∴ The equation represents a paraboloid and the reduced form is $12x^2+4y^2=4z$ *i. e.* $3x^2+y^2=z$.

II—Euler's Function ϕ (N)

If $p_1, p_2, \ldots p_n$ be the distinct prime factors of N, we shall prove the well-known relation

$$\phi(N) = N\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_n}\right)$$

by using two simple lemmas:

(i) $\phi(N) = p_1\,\phi\left(\frac{N}{p_1}\right)$ if $\frac{N}{p_1}$ contains p_1 as a factor

and (ii) $\phi(N) = (p_1-1)\,\phi\left(\frac{N}{p_1}\right)$ if $\frac{N}{p_1}$ is prime to p_1,

where $\phi(m)$ denotes the number of positive integers less than m which are relatively prime to m.

The positive integers less than N are given without repetition by $a \cdot \frac{N}{p_1} + r$ for $a = 0, 1, \ldots p_1 - 1$ and $r = 0, 1, \ldots\ldots\left(\frac{N}{p_1} - 1\right)$, $a = r = o$ being excepted.

Evidently $a\,\frac{N}{p_1} + r$ is prime to $\frac{N}{p_1}$ and therefore to N in case (i), if and only if r is prime to N/p_1. Since every such value of r can be associated with the p_1-values of a, there are $p_1\,\phi(N/p_1)$ less than N and prime to N.

Hence, the first lemma is proved.

In case (ii), if r_1 be any integer prime to N/p_1 and less than N/p_1 the set of integers

$$r_1,\ \frac{N}{p_1}+r_1,\ 2\,\frac{N}{p_1}+r_1,\ \cdots\ (p_1-1)\,\frac{N}{p_1}+r_1$$

is congruent to the set of integers $0, 1, \cdots (p_1 - 1)$ rearranged, (mod p_1) and includes exactly $(p_1 - 1)$ integers not divisible by p_1 and therefore prime to p_1.

Thus, in this case, we can associate only $(p_1 - 1)$ values of a with each r_1 belonging to $\phi(N/p_1)$, so as to obtain integers prime to N/p_1 and p_1 and therefore prime to N.

Hence $\phi(N) = (p_1 - 1)\,\phi\left(\frac{N}{p_1}\right)$.

If $N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, we get by repeated applications of lemma (i),

$$\phi(N) = p_1 \phi\left(p_1^{\alpha_1 - 1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}\right)$$
$$= p_1^2 \phi\left(p_1^{\alpha_1 - 2} p_2^{\alpha_2} \cdots p_n^{\alpha_n}\right)$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$= p^{\alpha_1 - 1} \phi\left(p_1, p_2^{\alpha 2} \cdots p_n^{\alpha_n}\right)$$

and by lemma (ii),

$$\phi\left(p_1 p_2^{\alpha_2} \cdots p_n^{\alpha_n}\right) = (p_1 - 1)\, \phi\left(p_2^{\alpha_2} \cdots p_n^{\alpha_n}\right)$$

Thus $\phi(N) = p_1^{\alpha_1 - 1}(p_1 - 1)\, \phi\left(p_2^{\alpha_2} \cdots p_n^{\alpha_n}\right)$

$$= p_1^{\alpha_1 - 1}(p_1 - 1)\, \phi\left(\frac{N}{p_1^{\alpha_1}}\right)$$

Repeating the same process with $\phi\left(\frac{N}{p_1^{\alpha_1}}\right), \phi\left(\frac{N}{p_1^{\alpha_1} p_2^{\alpha_2}}\right) \cdots$ and so on, we get

$$\phi(N) = p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \cdots p_n^{\alpha_n - 1}(p_1 - 1)(p_2 - 1) \cdots (p_n - 1)$$
$$= N\,(1 - p_1^{-1})(1 - p_2^{-1}) \cdots (1 - p_n^{-1}).$$

Thus the theorem is proved.

III—Homogeneous Products

We give below a short proof of the result ${}_nH_r = {}^{n+r-1}C_r$. As in Chrystals' Algebra, Part II, p 11, we add to each of the r combinations all the n-letters $a_1\ a_2\ a_3 \cdots a_n$ and get products of the form $a_1^{\alpha_1}\ a_2^{\alpha_2} \cdots\cdots a_n^{\alpha_n}$, where $\sum_{i=1}^{i=n} \alpha_i = n + r$

Consider the ordered set of $(n-1)$ numbers

$(\alpha_1,\ \alpha_1 + \alpha_2,\ \alpha_1 + \alpha_2 + \alpha_3, \cdots\cdots \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1})$ ... (A)

which has one-to-one correspondence with the ordered set

$(\alpha_1,\ \alpha_2, \cdots\cdots\cdots\cdots \alpha_n)$ ... (B)

On account of the one-one correspondences the number of ways of forming (B) is the same as the number of ways of forming (A) which is simply the number of ways of selecting $(n-1)$ distinct numbers all less than $(n+r)$ and ordering them in ascending magnitude, *viz.* ${}^{n+r-1}C_{n-1}$ or ${}^{n+r-1}C_r$

$$\therefore\ {}_nH_r = {}^{n+r-1}C_r.$$

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IV—On Vandermonde's Theorem

The following is a simple† proof of Vandermonde's theorem. It is seen here as a direct consequence of Leibnitz's theorem for the repeated differentiation of the product uv of two functions u and v of x.

Theorem.—

$$\sum_{s=0}^{r} {}_mC_{r-s}\ {}_nC_s = {}_{m+n}C_r$$

for all m and n and positive integral r.

Proof :—Applying Leibnitz's theorem with $u=x^m$ and $v=x^n$ we have

$$\sum_{s=0}^{s=r} {}_rC_s\ m(m-1)\cdots(m-r+s+1)x^{m-r+s}\ n(n-1)\cdots(n-s+1)\,x^{n-s}$$
$$=(m+n)\ (m+n-1)\cdots\cdots(m+n-r+1)\,x^{m+n-r}$$

On dividing both sides by x^{m+n-r},

$$\sum_{s=0}^{r} {}_rC_s\ \frac{m\,!}{(m-r+s)\,!}\ \frac{n\,!}{(n-s)\,!}=\frac{(m+n)\,!}{(m+n-r)\,!}$$

On dividing both sides by $r\,!$ and noting that

$${}_nC_r=\frac{n\,!}{(n-r)\,!\ r\,!}$$

we have the result.

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† Cf :—Cayley's Proof, *Messenger of Mathematics* Vol. V.

Andhra University, Waltair

K. SAMBASIVA RAO.

REVIEWS

ACTUALITES SCIENTIFIQUES ET INDUSTRIELLES (Hermann, Paris)

Colloque Consacré a la Théorie des Probabilités:

PREMIERE PARTIE: *Conférences d'introduction*: Introduction par M. R. Wavre, Allocution par M M. Frechet, Trois Conférences par M. M. Frechet, M. W. Heisenberg et M. G. Polya. Pp. 51. 11 fr. (1938) No. 734.

DEUXIEME PARTIE: *Les Fondements du Calcul des Probabilités*: par M. M. P. Cantelli, W. Feller, M. Frechet, R. De Mises. J. F. Steffensen et A. Wald. Pp. 99. 21 fr. (1938). No. 735.

TROISIEME PARTIE: *Les Sommes et les Fonctions de Variables Aléatoires*: par MM H. Cramér, P. Lévy et R. De Misés. Pp. 68. 20 fr. (1938). No. 736.

QUATRIEME PARTIE: *Le Principe Ergodique et les Probabilités en Chaîne*: par MM. E. Hopf, B. Hostinsky, O. Onicescu, V. Romanovsky Pp. 66. 18 fr (1938). No. 737.

CINQUIEME PARTIE: *Les Fonctions Aléatoires*: Par MM. S. Bernstein, E. Slutsky, A. Steinhaus. Pp. 73. 20 fr. (1938) No. 738.

SIXIEME PARTIE: *Conceptions Diverses*: Par MM. B. de Finetti, V. Glivenko, G. Neyman. Pp. 57. 11 fr. (1938) No. 739.

SEPTIEME PARTIE: *La Statistique Mathematique*: Par MM. E. Dodd, C. Jordan, N. Obrechkoff. Pp. 64. 18 fr. (1938) No. 740

The Theory of Probabilities is a branch of mathematics which is rapidly coming to the forefront as an essential tool of research in several fields of knowledge as widely separated as Quantum Theory and Business Economics. Almost every mathematician is transforming himself into a probabilist and channels are opened out to make the highest achievements of pure mathematics available for rearing this new edifice. International Conferences are called in for mathematical co-operation in the new architecture. In our own country, the Indian Science Congress was compelled to recognise Mathematics as a separate section, by virtue of its association with Statistics. The year 1937 has witnessed two conferences, one in each hemisphere, on the subject of Probability. The earlier conference took place at the Graduate School of the United States Department of Agriculture where J. Neyman spoke about the

modern view point on the classical Theory of Probability, his theories of confidence intervals and statistical estimation, and application to plant breeding, time-series analysis in economics and census by sampling. A few months later, in October 1937, there was a 'Colloquy' on 'Probabilities' organised by the University of Geneva. The original intention of the organisers was to have both 'Probabilities' and their 'applications'. But as the response to first invitations was full of probabilists, the idea of 'applications' was given up for the moment The memoirs consecrated to the pure theory are of such diverse character and so numerous that even the eminent president Maurice Frechet despairs of presenting a complete resumé of the situation. As the organisers had given entire freedom in the selection of subjects, it is perhaps not illegitimate to infer that the subjects presented here indicate the direction of movement in current researches. Usually the proceedings of Geneva Conferences are published in L'Enseignement mathematique (edited by M. Fehr) and other Swiss reviews. But as these reviews cannot accommodate more than 300 pages of matter at a time, it was thought fit to have recourse to the house of Hermann & Co for a rapid and early publication of the papers on 'probability' discussed at the conference. This is the genesis of the eight fascicules devoted to this colloquy. Seven of these are noticed here.

The titles of the fascicules give only a rough indication of the contents, the discrepancies in placement being probably due to convenience of printing. The first part is the most heterogeneous and the third least, while the sixth part has a vague heading 'diverse conceptions'. The papers may be re-ordered thus:

1. Les Fondements du Calcul des Probabilites; by M Frechet, W. Feller, R. de Mises, J. F. Steffensen, A. Wald and B. de Finetti.

2. Le Principe Ergodique et les Probabilites en Chaine, by G. Polya, E Hopf, B. Hostensky, O. Onicescu, V. Romanovsky and W. Heisenberg.

3. Les Sommes et les Fonctions de Variables Aléatoires by H. Cramer, P. Levy, R. de Mises.

4. Les Fonctions Aléatoires by S. Bernstein, E. Slutsky, A. Steinhaus, and V. Glivenko.

5. La Statistique Mathematique by G. Neyman, E. Dodd, C. Jordan, and N. Obrechkoff.

There are twenty-three papers in all, fully reported, one in English, three in German and the rest in French. The titles of the papers are given below (translated into English) in the order in which they appear in the fascicules.

M. Frechet: On the current principles in the evolution of recent researches on the Calculus of Probabilities.

G. Polya of Zurich. Random walk in a net-work of roads. (This was accompanied by a film demonstration).

W. Heisenberg of Leipzig: Probability notions in the Quantum theory of wave-fields.

P. Cantelli of Rome: On the definition of Contingent Variables (Only a short abstract appears here).

W. Feller of Stockholm: The axioms of the Calculus of Probabilities and their relation to experience.

M. Frechet of Sorbonne: Exposition and discussion of some recent researches on the fundamentals of the Calculus of Probabilities.

R. de Mises of Istanbul: Some remarks on the fundamentals of the Calculus of Probabilities.

J. F. Steffenson of Copenhagen: Frequency and Probability.

A. Wald of Vienna: On the validity of the notion of 'Collectives.'

P. Levy of Paris: The arithmetic of the laws of Probability and the finite products of the laws of Poisson.

R. de Mises: Generalisation of the classical limit theorems.

E. Hopf of Leipzig: The Statistical problem and its sequel in Classical Mechanics.

B. Hostinsky of Masaryk: Random fluctuations in the number of objects or points in a compartment.

O. Onicescu of Bucharest (Rumania): General theory of chains of complete connection (liaison).

V. Romanovsky of Tachkent: Some new problems in the theory of Markoff Chains.

S. Bernstein of Leningrad: Differential Stochastic Equations.

E. Slutsky of Moscow: On almost periodic random functions and the decomposition of stationary random functions into components.

H. Steinhaus of Lwow (Poland): Theory and applications of functions independent in the stochastic sense.

B. de Finetti of Trieste (Italy): On the condition of Partial Equivalence.

A. Kolmogoroff of Moscow: On random functions (laws of distribution in functional space) and their applications (This paper was not ready for publication here. It will be published in the series 'Exposes d' analyse generale' directed by M. Frechet in Actualites Scientifiques et Industrielles).

V. Glivenko of Moscow: On the law of large numbers in functional space.

J. Neyman of London: Statistical estimation treated as a classical problem of Probability.

E. Dodd of Texas, Austin (U. S. A.): Certain coefficients of regression or trend associated with largest likelihood.

C. Jordan of Budapest (Hungary): Criticism of Correlation from the probability point of view.

N. Obrechkoff of Sofia (Bulgaria): On the laws of Poisson, the series of Charlier and finite difference equations of the first order with constant coefficients.

From the above list of topics and their authors it is evident that we are entirely in a new world of probabilities little known to the English speaking people. We are more or less on Russian soil and its borders Scandinavia, Poland and the Balkan States, with Markoff's Chains elaborated by Onicescu and others, Kolmogoroff's functional spaces, Bernstein's differential stochastic equations, Mises's Collectives and Cramer's generalisation of the famous Laplace-Liapounoff limit-theorem. Two Germans speak on the new probability applications to Quantum theory and classical mechanics. The solitary voice of J. F. Steffensen champions forth the statistical definition of 'Probability' and does not see any reason for the several new theories of probability (each differing from the others in some detail adjusted to suit a particular pet view-point), as there is for the existence of one Geometry for Einstein and another for land-Surveyors.

There are two fundamental facts, which according to Henri Poincare characterise 'chance': (i) a very large number of small causes, each feeble in effect, acting simultaneously, and (ii) the existence of one or a small number of small causes, each producing a large effect. Gauss and Laplace have tackled mainly the first of the above two aspects, and Poincare had devised the method of arbitrary functions to deal with the latter. Poincare's method is extended by Hostinsky, Eberhard, Hopf, Copeland and de Mises. For a proper appreciation of these methods, it is necessary to consult the treatises of Borel, Castelnuvo, Khintchine, Kolmogoroff and Frechet. R. de Mises's lectures on 'Probability, Statistics and Truth' recently translated into English gives an excellent exposition of the theory of 'Collectives'. This theory has received further refinements from Popper, Reichenbach, Copeland, Wald and Ville. But S. Dumas and de Finetti who are both actuaries have not accepted the definition of de Mises. As an alternative to the 'Collective' theory, there is another characterised by Neyman as 'la definition classique modernisee', based on the modern theory of sets of points, sample space, and Lebesgue measure.

We are fairly familiar with Neyman's theory of statistical estimation as opposed to R. A. Fisher's, as they are accessible enough to English readers through the Statistical Research Memoirs of the Department of Statistics, University College, London and the American publication referred to in the earlier part of this review. This theory is adumbrated here with several new definitions, such as 'ensemble estimateur', 'ensembles d'acceptation' and 'ensembles estimateurs bien places' and theorems leading to the construction of 'intervals of confidence'. Neyman's paper may have therefore been

included more justifiably under 'La Statisque Mathematique' than under the vague title 'Conceptions Diverses'. Among the 'statistical applications' which are, of course, very few we find, to our great relief, a paper in English by E. Dodd on the nature of the coefficient of regression expressed as a more general mean than the weighted arithmetic mean of the ratios of variables y_i/x_i. The author derives several expressions with the help of R. A. Fisher's method of maximum likelihood applied to appropriate frequency distributions.

The notion of chains which was introduced in analysis by Markoff about three decades ago for the study of sequences of linked events appears now called on to play a fundamental role not only as a mathematical idea but as an instrument for the study of several natural phenomena. If $E_1, E_2, \cdots E_m$ be a finite aggregate of states in a system and p_{ik} the probability of passage from E_i to E_k, we have a simple chain. The infinite sequences of states $E_{i_1}, E_{i_2}, \cdots E_{i_n}, \cdots$ where i_n is simply one of the m given states will give rise to the infinite chain of probabilities $p_{i_1 i_2}, p_{i_2 i_3}, \cdots p_{i_{n-1} i_n} \cdots$ This scheme is applicable also to a continuous aggregate of states.

If $p^{(n)}_{ik}$ be the probability of passing by n successive changes from E_i to E_k, we get the recurrence formula

$$p_{ik}^{(n+1)} = \sum_{s=1}^{m} p_{is}\, p_{sk}^{(n)}$$

More generally,

$$p_{ik}^{(n+l)} = \sum_{s=1}^{m} p_{is}^{(n)}\, p_{sk}^{(l)} \qquad \ldots \quad (A)$$

Markoff has demonstrated that under sufficiently general circumstances, $\lim p_{ik}^{(n)} = p_k$, i.e. the limiting probability of the system which attains the state of E_k is independent of the initial state. The 'ergodic principle' is based on this result. One generalised form of the chain is

$$p_{i_1},\ p_{i_1 i_2},\ p_{i_1 i_2 i_3},\ \cdots$$

where the probability of any change depends on that of all of its predecessors. Such chains are to be found in the phenomena of heredity.

Onicescu constructs what he calls 'les chains à liasons completes' which has its statistical analogue in nature in the spread of a contagious disease like tuberculosis which is also hereditary. This chain obeys a composition law which generalises (A) in the form

$$p_s^{\,n}\ (x) = \sum_{w=1}^{m+1} x_k\ p_s^{(n-1)}\ [f_{w1}\,(x),\ f_{w2}\,(x),\ \cdots f_{km}\,(x)].$$

Regarding the fundamental formula (A), Onicescu remarks that differential equations, equations of propagation of waves, matrices in Hilbert's space and chains of Markoff are all governed by the same group based on (A). This puts in evidence the rigorous determinism which dominates the propagation of states in a material system.

We are tempted to quote Laplace's prophetic words: 'If one considers the analytic methods to which this theory (of probability) has given birth, the truth of the principles on which it is based, the fine and delicate logic which is demanded by their employment in the solution of problems, the matters of public importance which are dependent on them, and the extension which it has received and which it can yet receive by its applications to the most important questions of natural philosophy and moral science .. . . one sees that there is no science more worthy of our meditations, and that it would be most useful to introduce it into our system of public instruction.' We have nothing to add to Laplace's recommendation to introduce the subject seriously in our Honours Courses and post-graduate studies. The publications like those before us will be an unfailing source of inspiration to the interrogators of 'L'urne.'

We wish the *Makers of Probability* all success.

A. A. K.

ANNOUNCEMENTS AND NEWS

The following gentlemen have been elected members of the Society :—

V. P Venkatachari Esq., M.A., Oosmania University, Hyderabad, (Deccan).

H. Subramani Iyer Esq., M.A., Ph D., Travancore University, Trivandrum.

P. C. Mahalanobis Esq, M.A., B.SC. I. E. S., Head of the Statistical Laboratory, Presidency College, Calcutta.

The Librarian wishes to announce that owing to the International situation (for which he hints he is not to be blamed) the number of available current mathematical periodicals has diminished very considerably. He invites the cooperation of all members in preventing stagnation in the circulatory system.

Dr. Hansraj Gupta of Government College, Hoshiarpur, is collecting notes relating to the lives and work of mathematicians engaged in research in India with a view to the publication of this material later. He requests their co-operation in the work by sending him 2 copies of their photographs, a short sketch of their lives, a complete list of their published papers with the fullest details and short summaries of the most important papers.

Sri S. Minakshisundaram, M.A. has been appointed Research Assistant in Topology in the Madras University to help Dr. Vaidyanathaswami in giving guidance to research students.

SOLUTIONS TO QUESTIONS

Question 1597

(R. RAMANUJAN): Given four line segments measuring 4, 9, 12 and 21 units respectively, find geometrically the values of x and y satisfying the equations

$$\frac{xy\,(x+y+8)}{21\,(x+y)} - \frac{9\,(x+y)}{25\,(x+y-8)} = (x^2-y^2)^2 - 63\,(2x^2+2y^2-91) = 0.$$

Solution by the Proposer

The second equation simplifies into

$$x^4+y^4+8^4-2x^2y^2-2x^2{\cdot}8^2-2y^2{\cdot}8^2+64\times 27=0$$

which can be written

$$\left\{\frac{x+y+8}{2}\cdot\frac{x-y+8}{2}\cdot\frac{-x+y+8}{2}\cdot\frac{x+y-8}{2}\right\}^{\frac{1}{2}} = \sqrt{108} \qquad \ldots (1)$$

the first equation becomes

$$\left\{xy\left[1-\frac{8^2}{(x+y)^2}\right]\right\}^{\frac{1}{2}} = \frac{\sqrt{(189)}}{5} \qquad \ldots (2)$$

If x, y, 8 are the sides of triangle, the left side of (1) gives the area and that of (2) gives the length of the internal bisector of the vertical angle contained by the sides x and y. If a triangle with sides x, y, 8 and the left side of (2) for the length of the external bisector be drawn then $(x, -y)$ and $(-x, y)$ will be solutions of the given equations. These considerations give the following geometric method of solving the given equations :

Construct a triangle satisfying the following condition :—(i) base = 8 units (ii) area = $\sqrt{(108)}$ units (iii) the length of the internal bisector of the vertical angle equals $\frac{\sqrt{(189)}}{5}$ units. Let x_1 and y_1 be the lengths of the remaining sides. Similarly construct another triangle satisfying (i), (ii) and (iii), the external bisector of the vertical angle being of length $\frac{\sqrt{(189)}}{5}$ units. Let x_2 and y_2 be the lengths of the remaining two sides. Then the four solutions of the equations are (x_1, y_1), and $(-x_1, -y_1)$ (x_2, y_2) and $(-x_2, y_2)$. That the above construction can be carried out by using the 4 segments of given lengths and a pair of compasses is a matter of detail which can be verified.

Question 1668

(ALFRED MÖSSNER) :—We have

$$17^n + 17^n = (-3)^n + 5^n + 12^n + 20^n \qquad (n = 1,\ 2,\ 3)$$

Is the equation $A_1^n + A_2^{2n} = B_1^n + B_2^n + B_3^n \qquad (n = 1,\ 2,\ 3)$
also solvable when $A_1 \neq A_2$? What is the general solution?

(ii) We have $7^n + 7^n = 3^n + 5^n + 8^n \qquad (n = 2,\ 4)$

Is the equation $E_1^n + E_2^n = F_1^n + F_2^n + F_3^n \qquad (n = 2,\ 4)$
also solvable for $E_1 \neq E_2$?

Remarks by A. Weiland

The following is a solution when n takes only the values 1 and 3. The identity

$$(t^2 - 2t - 1)^k + (t - 1)^{2k} = 2^k + (t^2 - 3t)^k + (t^2 - t - 2)^k$$

is easily verified to be true for $k = 1$ and 3. For $t > 3$ all the expressions involved are positive. Thus we get for

$$t = 4, \quad \left.\begin{array}{l} 7^3 + 9^3 = 2^3 + 4^3 + 10^3 \\ 7 + 9 = 2 + 4 + 10. \end{array}\right\};$$

$$t = 5, \quad \left.\begin{array}{l} 14^3 + 16^3 = 2^3 + 10^3 + 18^3 \\ 14 + 16 = 2 + 10 + 18. \end{array}\right\};$$

$$t = 6, \quad \left.\begin{array}{l} 23^3 + 25^3 = 2^3 + 18^3 + 28^3 \\ 23 + 15 - 2 + 18 + 28. \end{array}\right\};$$

$$t = 7, \quad \left.\begin{array}{l} 34^3 + 36^3 = 2^3 + 28^3 + 40^3 \\ 34 + 36 = 2 + 28 + 40. \end{array}\right\}.$$

Question 1775

(A. A. KRISHNASWAMY AYYANGAR).—Prove the following results

$$\frac{\pi}{\sqrt{12}} = 1 - \frac{1}{3^2} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \cdots\cdots$$

$$\frac{\pi}{16} = \frac{1}{1^5 + 4 \cdot 1} - \frac{1}{3^5 + 4 \cdot 3} + \frac{1}{5^5 + 4 \cdot 5} - \cdots\cdots$$

$$\frac{\pi - 3}{4} = \frac{1}{3^3 - 3} - \frac{1}{5^3 - 5} + \frac{1}{7^3 - 7} - \cdots\cdots$$

$$\frac{4 - \pi}{8} = \frac{1}{4^2 - 1} + \frac{1}{8^2 - 1} + \frac{1}{12^2 - 1} + \cdots\cdots$$

$$\frac{\pi - 3}{6} = \frac{1}{(2 \cdot 2^2 - 1)^2 - 2^2} + \frac{1}{(2 \cdot 4^2 - 1)^2 - 4^2} + \frac{1}{(2 \cdot 6^2 - 1)^2 - 6^2} + \cdots$$

Solution by C. B. Rathie.

1. $1 - \frac{1}{3^2} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3}$

$$= \sqrt{3}\left\{\frac{1}{\sqrt{3}} - \frac{1}{3}\cdot\left(\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{5}\cdot\left(\frac{1}{\sqrt{3}}\right)^5 - \frac{1}{7}\left(\frac{1}{\sqrt{3}}\right)^7 + \cdots\right\}$$

$= \sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$, by applying Gregory's Series,

$$= \sqrt{3}\cdot\frac{\pi}{6} = \frac{\pi}{\sqrt{12}}.$$

The above series is identical with that of Ex. 6, page 112, of Loney's Trigonometry Part II.

2. $\frac{1}{1^5 + 4\cdot 1} - \frac{1}{3^5 + 4\cdot 3} + \frac{1}{5^5 + 4\cdot 5} - \cdots\cdots$

We have $\frac{1}{x^5 + 4x} = \frac{1}{4}\left\{\frac{1}{x} - \frac{x^3}{x^4 + 4}\right\}$.

It is easily seen that $\frac{x^3}{x^4 + 4} = \frac{1}{2}\left\{\frac{x-1}{(x-1)^2 + 1} + \frac{x+1}{(x+1)^2 + 1}\right\}$.

Therefore, $\frac{1}{x^5 + 4x} = \frac{1}{4}\cdot\frac{1}{x} - \frac{1}{8}\left\{\frac{x-1}{(x-1)^2 + 1} + \frac{x+1}{(x+1)^2 + 1}\right\}$.

Putting $x = 1, 3, 5, \cdots$, we get

$$\frac{1}{1^5 + 4\cdot 1} = \frac{1}{4}\cdot\frac{1}{1} - \frac{1}{8}\left\{0 + \frac{2}{2^2 + 1}\right\},$$

$$-\frac{1}{3^5 + 4\cdot 3} = -\frac{1}{4}\cdot\frac{1}{3} + \frac{1}{8}\left\{\frac{2}{2^2 + 1} + \frac{4}{4^2 + 1}\right\},$$

$$\frac{1}{5^5 + 4\cdot 5} = \frac{1}{4}\cdot\frac{1}{5} - \frac{1}{8}\left\{\frac{4}{4^2 + 1} + \frac{6}{6^2 + 1}\right\},$$

Hence, on adding, the given series

$$= \frac{1}{4}\left\{1 - \frac{1}{3} + \frac{1}{5} - \cdots\right\} + 0 = \frac{\pi}{16}.$$

3. $\frac{1}{3^3 - 3} - \frac{1}{5^3 - 5} + \frac{1}{7^3 - 7} - \cdots\cdots$

$$= \frac{1}{3}\cdot\frac{1}{3^2 - 1} - \frac{1}{5}\cdot\frac{1}{5^2 - 1} + \frac{1}{7}\cdot\frac{1}{7^2 - 1} - \cdots\cdots$$

$$= \frac{1}{3}\cdot\frac{1}{2\cdot 4} - \frac{1}{5}\cdot\frac{1}{4\cdot 6} + \frac{1}{7}\cdot\frac{1}{6\cdot 8} - \cdots\cdots$$

$$= \frac{1}{2}\left\{\frac{1}{3}\left(\frac{1}{2} - \frac{1}{4}\right) - \frac{1}{5}\left(\frac{1}{4} - \frac{1}{6}\right) + \frac{1}{7}\left(\frac{1}{6} - \frac{1}{8}\right) - \cdots\cdots\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{2\cdot 3} - \frac{1}{3\cdot 4} - \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} + \frac{1}{6\cdot 7} - \frac{1}{7\cdot 8} - \cdots\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{2} - \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \frac{1}{6} - \frac{1}{7} - \frac{1}{7} + \frac{1}{8} - \cdots\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{2} - 2\left(\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \cdots\right)\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{2} - 2\left(1 - \tan^{-1} 1\right)\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{2} - 2 + \frac{\pi}{2}\right\} = \frac{\pi - 3}{4}.$$

4. $\dfrac{1}{4^2 - 1} + \dfrac{1}{8^2 - 1} + \dfrac{1}{12^2 - 1} + \cdots$

$$= \frac{1}{3\cdot 5} + \frac{1}{7\cdot 9} + \frac{1}{11\cdot 13} + \cdots$$

$$= \frac{1}{2}\left\{\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \frac{1}{13} + \cdots\right\}$$

$$= \frac{1}{2}\left\{1 - \tan^{-1} 1\right\} = \frac{1}{2}\left(1 - \frac{\pi}{4}\right) = \frac{4 - \pi}{8}.$$

5. $\dfrac{1}{(2\cdot 2^2 - 1)^2 - 2^2} + \dfrac{1}{(2\cdot 4^2 - 1)^2 - 4^2} + \dfrac{1}{(2\cdot 6^2 - 1)^2 - 6^2} + \cdots$

Now $\dfrac{1}{(2x - 1)^2 - x} = \dfrac{1}{4x^2 - 5x + 1} = \dfrac{1}{(x - 1)(4x - 1)}$

$$= \frac{1}{3}\left\{\frac{1}{x - 1} - \frac{4}{4x - 1}\right\}.$$

Putting $x = 2^2, 4^2, 6^2, \cdots$

$$\frac{1}{(2\cdot 2^2 - 1)^2 - 2^2} = \frac{1}{3}\left\{\frac{1}{2^2 - 1} - \frac{4}{4\cdot 2^2 - 1}\right\} = \frac{1}{3}\left\{\frac{1}{1\cdot 3} - \frac{4}{4^2 - 1}\right\},$$

$$\frac{1}{(2\cdot 4^2 - 1)^2 - 4^2} = \frac{1}{3}\left\{\frac{1}{4^2 - 1} - \frac{4}{4\cdot 4^2 - 1}\right\} = \frac{1}{3}\left\{\frac{1}{3\cdot 5} - \frac{4}{8^2 - 1}\right\},$$

$$\frac{1}{(2\cdot 6^2 - 1)^2 - 6^2} = \frac{1}{3}\left\{\frac{1}{6^2 - 1} - \frac{4}{4\cdot 6^2 - 1}\right\} = \frac{1}{3}\left\{\frac{1}{5\cdot 7} - \frac{4}{12^2 - 1}\right\},$$

. . . .

Hence on adding, the series becomes

$$= \frac{1}{3}\left\{\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots\right\} - \frac{4}{3}\left(\frac{1}{4^2 - 1} + \frac{1}{8^2 - 1} + \frac{1}{12^2 - 1}\right)\cdots$$

$$= \frac{1}{3}\left\{\frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \cdots\right)\right\} - \frac{4}{3}\cdot\frac{4 - \pi}{8}, \text{ by result 4.}$$

$$= \frac{1}{6} - \frac{4 - \pi}{6} = \frac{\pi - 3}{6}.$$

QUESTIONS FOR SOLUTION

1788. (A. E. PALKAR). Show that the number $2^{9961452}$ leaves the remainder 137438429185 when divided by $(524287)^2$.

1789. (P. KESAVA MENON). If (r, n) denotes the greatest common measure of r and n, and $\phi(n)$ is the number of integers less than n and prime to it and if $n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$, the p's being different prime factors, show that

$$\sum_{1 \leq r_1, r_2, \cdots r_s \leq n} (r_1 r_2 \cdots r_s, n) = \phi^s(n) \prod_{q=1}^{t} \binom{s+a_q}{r+a_q} (p_q - 1)^{-r}$$

$$= n^s \prod_{q=1}^{t} \frac{a_q(a_q+1) \cdots (a_q+s)}{s!} \prod_{q=1}^{t} \sum_{r=0}^{s} (-1)^r \frac{\binom{s}{r}}{p_q^r (a_q + r)}.$$

1790. (A. A. KRISHNASWAMI AYYANGAR). Elucidate from considerations of symmetry (without induction) that the symmetric function

$$\sum \alpha_1^{p1} \alpha_2^{p2} \cdots \alpha_n^{pn}$$

can be expressed in terms of power-sums

$$S_p \equiv \alpha_1^p + \alpha_2^p + \cdots + \alpha_r^p \qquad (r > n)$$

in the form

$$\sum A_{p'q'} \cdots S_{p'} S_{q'} \cdots \cdots$$

where $\quad p' + q' + \cdots = p_1 + p_2 + \cdots + p_n$

and $\quad p', q', \cdots$ are ths sums of different sets of the indices $p_1, p_2, \cdots p_n$, the summation extending over all the possible values of $p', q', \cdots$ under the circumstances.

Prove that $A_{p1q'} \cdots$ depends only on the number of the integers $p', q', \cdots$ and the number of indices (p_1, p_2, p_n) that go to form $p', q', \cdots$.

Hence deduce the value of $A_{p1q'} \cdots$ by putting

$$p_1 = p_2 = \cdots = p_n = 1.$$

1791. (P. KESAVA MENON). If through any point O three secants are drawn to meet a conic in P, P^1; Q, Q^1; R, R^1 respectively then

$$\frac{\Delta\ PQR}{OP \cdot OQ \cdot OR} + \frac{\Delta\ P^1Q^1R^1}{OP^1 \cdot OQ^1 O \cdot R^1} = 0.$$

1792. (N. A. COURT, Oklahoma, U. S. A.). Find a sphere such that the polar planes with respect to this sphere of two given points shall pass respectively through two given lines.